Pumping Lemma

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Introduction

- A regular language can be defined by an FA/FSM

- Is there a way for us to effectively show that a language is not regular?
Pumping Lemma

• The pumping lemma is useful for disproving the regularity of a specific language.

• The Pumping lemma basically states that all sufficiently long strings \( (S) \) in a regular language \( (A) \) may be "pumped" – that is,
  • have a middle section of the string \( (y^n) \) repeated in an arbitrary number of times \( (I \text{ often use } 2 \text{ as my arbitrary } \#) \) – to produce a new string \( (S^1) \) that is also a part of the language.
Pumping Lemma

• If A is a regular language, then A can be pumped.

• A has a pumping length (some constant, P) such that any string S (where S>=P) may be divided into 3 parts (x, y, & z)

• S = x y z (3 substrings, x, y, z, with y being non-empty, such that the strings constructed by repeating y, 0 or more times, are still in A).
  • |xy| <=P (i.e. the length of x followed by y must be less or equal to the pumping length, P, thus, imposing a limit on how S may be spilt)

In addition:
• Any string, $S^1$, constructed from x $y^n$z (where n>=0) is also an element of A
• y cannot be lambda (i.e. y must be non-empty)
Proof by Contradiction

- Assume the language $A$ is regular
- $A$ has to have a pumping length: $P$
- (Since $A$ is regular) Any strings, $S$, in language $A$, that is longer than $P$, can be pumped
- Find $S$ | $S$ $\geq$ $P$
- Divide $S$ into $xyz$, where $|xy|$ $\leq$ $P$
- Consider the ways that $S$ can be divided into $x$ $y$ $z$

- Consider $S^1$, which is also an element of language $A$

(contradiction: Show that the necessary conditions cannot be satisfied at the same time)

- Show $S' = x y^n z$ (where $n$ $\geq$ $0$) cannot be "pumped" [remember we need to show only one counterexample to disprove "all"]
- Conclude that $S'$ cannot be pumped, because it is not a regular language. $S'$ is an element of language $A$ - thus $A$ is also not a regular language
Pumping Lemma

Example 1: Language A = \{a^n b^n | n \geq 0 \}

Before we get started, consider:

- This language has any number of a’s followed by the same number of b’s
- Consider that we need to keep a count of a’s to determine how many b’s - but FA cannot keep count of anything. FA cannot be used to define this A. (Note: Regular languages can be defined by FA’s)

- Keep the following approach in mind as we go through the examples:
  1. For the Proof by Contradiction, suppose A is a regular language
  2. We need a pumping length, P | any string S, in language A, is greater than or equal to P (S \geq P)
  3. Let’s use pumping length, P = 7 and let’s assign string S = a^p b^p
  4. Consider the ways that S can be divided into xyz, where |xy| \leq P
  5. Now consider S’ = xy^n z, where n \geq 0
  6. Show that S’ cannot be "pumped." [we need to show only ONE counterexample to disprove "all"]
  7. Conclude: String S’ is in language A. If S’ cannot be "pumped" then A cannot be "pumped," and therefore, A is not a regular language
Example 1: Use the pumping Lemma to prove, language \( A = \left\{ a^n b^n \mid n \geq 0 \right\} \) is not regular

[This language has any number of a’s followed by the same number of b’s]

Proof by Contradiction

- Suppose \( A \) is a regular language
- \( S \) is an element of \( A \) (i.e. \( S \) is a word in the language \( A \))
- Let pumping length, \( P \) be 7
- Let's assign \( n = 6 \), so string \( S = a^6 b^6 \)
- Consider the ways that \( S \) can be divided into \( xyz \), where \(|xy| \leq P\)

- Now consider \( S' = xy^m z \), where \( m = 2 \)
- Contradiction: Show that \( S' \) cannot be "pumped."
- Conclude: String \( S' \) is in language \( A \). If \( S' \) cannot be "pumped" then \( A \) cannot be "pumped," and therefore, \( A \) is not a regular language
. Pumping Lemma

Example 1: Continued
Pumping Lemma

Example 2: Use the pumping Lemma to prove, language $A = \{y y \mid y \in (0^n 1)\}$ is not regular
[This language has any number of 0's followed by 1, and then repeated once more (y y)]

Proof by Contradiction

- Suppose $A$ is a regular language
- $S$ is an element of $A$ (i.e. $S$ is a word in the language $A$)
- Let pumping length, $P$ be 7
- Let's assign $n = 7$, so string $S = 0^7 1$
- Consider the ways that $S$ can be divided into $xyz$, where $|xy| \leq P$

- Now consider $S' = xy^mz$, where $m = 2$
- Show that $S'$ cannot be "pumped." (in a regular manner) [we need to show only ONE counterexample to disprove "all"]
- Conclude: String $S'$ is in language $A$. If $S'$ cannot be "pumped" then $A$ cannot be "pumped," and therefore, $A$ is not a regular language
. Pumping Lemma

Example 2: Continued

(Consider the ways that $S'$ can be divided into $xy^nz$ and show that $S'$ cannot be "pumped" (let's assign $n=2$).

String $S'$ is in language $A$. If $S'$ cannot be "pumped" then $A$ cannot be "pumped," and therefore, $A$ is not a regular language.)