PCTM Magazine

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Volume LII, Number 1, Fall 2013

Inside

- Which coupon should you clip?
- Arches, windmills, and other structures
- This adder is not venomous

President's Message

"Mike's Long and Winding Road"

Dear Math Ed Friends and Family…

We are all well into another school year and doing what we love doing, teaching math. As I read and listen to the news, stories abound about scores on math assessments. Those stories make me ponder the following question: have we lost our sense of what it means to teach mathematics, or simply to teach, in favor of assessment?

As I laugh and share with math teaching friends, I often hear those individuals speak about the lack of engagement in their students and I cannot help but think that the focus on assessment is a factor in the lack of engagement. I realize that assessment is a critical part of the teaching process, but if that is the primary focus, there is a high probability that our practice lacks engagement.

I used to spend Sunday nights watching Desperate Housewives and wondering what are Bree, Susan, Gabby, and Lynette going to do next. The show was riveting and engaging… at least for me. So as we approach 2014, think about how you can bring that riveting engagement back to your math classroom, while being mindful of but not totally focused on assessment. Check out the multitude of resources available, including the Standards for Mathematical Practice, and ramp up your level of engagement. Let's live our 2013 conference theme in 2014 and spring some life back into our math classrooms!

Yours in teaching math,

Mike

Cover photo: A tessellation is a covering of an area using the repetition of a shape such as a hexagon. Quilting is an area where this mathematics has importance in the world outside of school. After all, if the shapes don't cover the quilt, it won't keep you very warm!

PCTM Magazine

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Volume LII, Number 1, Fall 2013

PCTM Magazine is published by the Pennsylvania Council of Teachers of Mathematics, an affiliate of the National Council of Teachers of Mathematics.

Articles and announcements for *PCTM Magazine*, an editor-reviewed publication, should be submitted electronically to the editors. For guidelines on manuscript submission, see **http://pctm.squarespace.com/pctm-magazine/**

The (outgoing) *PCTM Magazine* Editors are: Dave Kennedy, Shippensburg University (dikenn@ship.edu) and Randy Schaeffer, Kutztown University (schaeffe@kutztown.edu)

The (incoming) *PCTM Magazine* Editors are Cynthia Taylor and Tyrone Washington, who may be reached at **pctm.editor@gmail.com**

From the Editors

Dave Kennedy
Shippensburg Univ.

Randy Schaeffer
Kutztown Univ.

A Farewell to Paper

As you already realize if you are reading this issue, *PCTM Magazine* has gone online. As incoming co-editors Cynthia Taylor and Tyrone Washington stand ready to move the magazine into the digital age, Randy and I would like to thank numerous people who have helped us over the years.

First and foremost, thank you to all who contributed articles, news items, conference information, contest lists, president's messages, photos, and more. Needless to say, a magazine needs content, and we have been fortunate to have thoughtful and productive authors. We always valued your patience and understanding. Thanks also to everyone who wrote or emailed their notes of appreciation for the magazine.

Randy originated many of the thoughts above, and I shift now to some campus-specific thoughts. By way of thanking those at Shippensburg University who helped in critical ways, I thought it might be fun to describe the process of putting out a paper issue of the magazine; historians take notice!

The first job in creating a magazine was of course to edit and lay out the articles and features. I would then meet with Donna Jones in the Publications Office. Donna would help work out any bugs with our fonts, page-number icons, and the like, and would even Photoshop the cover photo to make it look better than it looked when she received it. Her superior command of Adobe InDesign software was always relied upon to straighten out some of my guesswork. Thanks to Donna for the many hours of work she gave to make the magazine look its best.

From Publications, the magazine would go to the print shop on campus, where Joe Amsler oversaw the printing, stapling, and boxing of over 1,000 copies. He was instrumental in recommending printing materials based on his knowledge of the capabilities of his equipment. With black-and-white interior pages and a color cover on special card stock, Joe had to keep everything coordinated so that each copy fit together logically. (Did you ever notice that your paper copies always had a multiple of four pages?) He did this flawlessly, and deserves thanks.

While the magazine was in press, it was time to mobilize for bulk mailing. The PCTM Secretary, Cathie Cooper in recent times and Barb MacDonald before that, painstakingly maintained an up-to-date file of names and addresses of all PCTM members — again, thanks for this work! — and would send the latest list at this point in the process. In recent years the list would be forwarded to Bob Witter in the Accounts Payable office at Shippensburg, and Bob would "certify the database," a beautiful process during which all addresses were checked against the U.S. Postal Service's records. If an address was no longer in service and/or had changed, this change would be picked up by Bob's software so that the magazine would be sent to the right place. Thanks to Bob and to Cathie, very few issues ever needed to be returned to sender.

After receiving the certified list back from Bob, I could do a mail merge and print the mailing labels, but not before Denny Starliper, the Clerical Supervisor in Central Receiving/Mail Services, arranged the names in a special order. To be acceptable to the postal service, the magazines had to be organized in a very particular way, typically into nine sacks, each containing many bundles. Denny would produce the ordered list showing whose magazine went into which bundle. He also

performed a useful bit of math by weighing ten copies of the magazine and dividing by ten to get the exact weight of an issue, which factored into the computation of mailing costs.

Once printed, the magazines were delivered to the Math Department for sorting and labeling. I will always be grateful to Math Department Secretary Pam McLaughlin for her willingness to help me with the process of rubber-banding the magazines into bundles of five to forty, and placing each bundle into the correct sack. Sometimes Pam recruited a student worker to help, which was also appreciated. Colleague Tom Evitts joined in on more than one occasion, so that what would have been a five-hour process passed much more quickly. At last the sacks would be ready for pickup. Denny Starliper would always have them whisked out of the math office and downtown to the Shippensburg P.O. in good time.

The printing and database certification have costs associated with them, and the respective Shippensburg personnel were always glad to communicate with PCTM Treasurer Steve Cicioni, who is automatic in his efficiency in distributing payments and keeping PCTM's books balanced. Steve would tell me that the production costs of the magazine were relatively low, and I know that much of the credit goes to all of these helpers who boosted our productivity.

To all who helped with any part of the magazine production, Randy and I offer heartfelt thanks. As Randy prepared his thoughts for this piece and communicated them to me, he wished to thank me for carrying on when he was sometimes unable to. I wish to thank him for the huge infusion of technical knowhow he brought to this position, and also for his wisdom in making editorial decisions. Finally, Randy emphasized something we were both thinking: to the new editors, best wishes!

---Dave (with help as usual from Randy)

Who am I?

Answer to the Spring 2013 *Who am I?* Danica McKellar

[*Editor's note: Danica starred in the movie* Flatland 2: Sphereland*, reviewed in this issue.*]

[*Editor's second note: Unlike previous Who am I? answers, this one is provided rightside-up for your computer viewing convenience.*]

Bed, Math, and Beyond

Alicia Myers, Hagerstown Community College

The BED MATH & BEYOND lesson is well suited for students being introduced to the concepts of percentages and inequalities. Students often struggle with the comprehension of inequalities, but when given relatable "real world" scenarios can easily make the connection and create a memorable learning experience. The basis of this activity is two coupons offered from BED BATH & BEYOND™. The first coupon gives 20% off any one item in the entire store, while the other coupon is $5.00 off any item priced at $15.00 or more. The goal is to have students investigate the prices of random items and formulate an inequality that would allow a customer to decide which coupon to use in any given situation.

For this activity, students are to go to the BED BATH & BEYOND™ website and select any item that is priced more than $15.00. Students select which coupon they believe will be the most beneficial to use at the checkout, prove algebraically that they are saving more money, and support their decision by explaining their reasoning. Having students select their own item allows for a variety of prices and answers. The diversity of solutions prompts a class discussion concerning the circumstances in which one coupon is more beneficial than the other.

An assortment of items is presented on the activity sheet (see question 3) in which students need to determine which coupon will save them the most money. This analysis guides students to understand prices where the coupons are advantageous. After discounting 20%, students compare what they saved versus using the $5-off coupon. Take note of the Banzai® Water Slide that is priced at $12.40. Often students look to use the $5-off coupon before realizing that a stipulation in using this coupon is that the item must be $15 or more; this continues the association with inequalities. The bath rug priced at $25.00 allows students to find the equivalent point between coupons; this too allows for an intriguing class discussion. Often students debate that the $5 off coupon is more logical to use because you can save your 20%-off coupon for a more costly item in the future. This discussion leads up to the next question: At what item price would the discount using a 20%-off coupon be equal to the discount using the $5.00-off coupon? How can you represent this algebraically? Typically students use the guess-and-check method to find that the coupons are equivalent at $25.00, but supporting their answer algebraically with reasoning and proof is often found to be challenging. After analyzing the scenario, students collectively realize that they are searching for a price x in which the $5 is equivalent to 20% of that price. From this realization, students find it easier to formulate the equation $5 = .2x$ and solve for x.

Throughout the activity students familiarize themselves with the percentage concept through adding sales tax, discounting coupons and finding the original price of an item that was bought on sale. These problems guide students into understanding the foundation of percentages while teaching them life skills that they will use on a daily basis.

[See pages 7-8 for the activity sheet that accompanies this article.]

BED MATH & BEYOND

Name:

BED BATH & BEYOND™ offers two types of coupons. The first coupon gives you 20% off any item in the store. The second coupon is $5.00 off any item priced at $15.00 or more. You are to go to the store website www.bedbathandbeyond.com any choose any item of your choice that is more than $15.00. Record the product name and price then answer the following questions.

Product Name:_____Product Price:_____

1. Which coupon would you use at the checkout? Support your answer algebraically and provide reasoning. (What will your total be with the 20%-off coupon versus the $5.00-off coupon?)

2. Once you select the appropriate coupon you are ready to check out. The cashier gives you your discount and charges 6% sales tax.

   How much sales tax must you pay?

   What is your total bill?

3. Which coupon would you use for the following four items?

Nostalgia Electrics™ Mini Stainless Steel Electric Fondue Pot

$19.99

Keurig® Vue® Brewer V500 Single Cup Home Brewing System

$169.99

Banzai® 3-D Shark Bite Water Slide

$12.49

Hang Ten Bath Rug

$25.00

4. At what item price would the discount of using a 20%-off coupon be equivalent to using the $5.00-off coupon? How can this be represented through an equation?

5. You paid $170.00 for a Keurig® Coffee maker during a 30%-off sale. What was the regular price?

6. Rachel Ray Cookware™ has a regular price of $180.00. BED BATH & BEYOND™ is having a 30%-off sale and you decide to use your 20%-off coupon. How much will you save? What will you pay for the cookware?

7. Write 3 inequality statements for the coupon scenario. Let P be the price of the item.

If $\boldsymbol{P} < \$\mathbf{15.00}$ then you should use the <u>20%-off any item coupon</u>

If _____ $\leq \boldsymbol{P} \leq$ _____then you should use the_____

If $\boldsymbol{P} >$ _____ then you should use the_____

Factoring Trinomials Using the ARCH Method

Cynthia E. Taylor, Millersville University of Pennsylvania
Dorothee Blum, Millersville University of Pennsylvania
Cathy Falci, Hempfield High School

Factoring polynomials is an essential algebraic skill that students should acquire during their Algebra I and Algebra II courses. This skill is used in many situations in later mathematics courses such as pre-calculus and calculus where students are expected to be proficient in factoring. Students who plan to pursue careers in science, business, and mathematics need these higher-level mathematics courses and cannot afford to have the door of opportunity shut just because they have not mastered the skills and language of algebra. The importance of factoring is supported in the Pennsylvania Keystone Exams: Algebra II Assessment Anchors and Eligible Content (PDE, 2012) where it is identified in several eligible content descriptions aligning to the Pennsylvania Anchors. However, high-school teachers know that many students struggle to acquire this skill especially as they move from factoring trinomials of the form

$$x^2 + bx + c$$

to trinomials of the form

$$ax^2 + bx + c, \text{where } a \neq 1$$

.

Many schemes have been developed over the years to help students learn how to factor non-monic trinomials. Two of these are the "rainbow method," based on factoring by grouping, and the "slide and divide method." Another method that appears to be a hybrid of these two methods is articulated below. We will refer to this method as the "ARCH method."

To factor the trinomial

$$5x^2 - 14x + 8$$

using the ARCH method, students would approach this problem in a manner similar to the way they would when factoring monic trinomials. However, the first step should be to factor out all common numerical factors and then think of two factors of 5(8) = 40 (see row 1) whose sum is -14 (see row 2). In this case, the factors would be -10 and -4. Next, the students would "divide" each of these two factors by *5x*, the product of the leading coefficient of the trinomial and the variable *x* (see row 3) and simplify each "ratio" (see row 4). Finally, they would rewrite the two "ratios" in the form of two factors (see row 5). These factors are what the students are trying to determine.

Factor: $5x^2 - 14x + 8$

row 1		40		
row 2		-10	-4	
row 3		$\frac{-10}{5x}$	$\frac{-4}{5x}$	
row 4	$\frac{-2}{x}$			$\frac{-4}{5x}$
row 5	(x-2)			(5x-4)

Figure 1. An example of factoring a trinomial using the ARCH method

It should be noted that the formation of the two "ratios" is not a division problem, but simply a mechanism that organizes the students' work.

After trying a few examples, it becomes clear that this technique seems to work every time, as long as the greatest common divisor of the three coefficients is 1. The question now is, mathematically, why does the ARCH method work? Let's consider the general expression

$$ax^2 + bx + c$$

and the steps described above. First, we need to find the product *ac* and then find its prime factorization:

$$ac = p_1 p_2 \dots p_t \text{ where each } p_i \text{ is prime.}$$

We need two factors *m* and *n* of the product *ac* such that $m + n = b$. Since $mn = ac$, the product *mn* has the same prime factorization as *ac*. By rearranging these prime factors, we find integers *r, s, u, v* so that

$a = rs$, $c = uv$, $m = ru$, and $n = sv$.

Applying the ARCH method to the general case, we have:

row 1	ac	
row 3	$\frac{m}{ax}$	$\frac{n}{ax}$

By substitution and simplification:

row 3	$\frac{ru}{rsx}$	$\frac{sv}{rsx}$
row 4	$\frac{u}{sx}$	$\frac{v}{rx}$

Thus, the factors are (sx + u)(rx + v). If these two binomials are multiplied, we obtain:

$$(sx + u)(rx + v) = rsx^2 + urx + svx + uv$$

By substitution and further simplification, we then have:

$$\begin{aligned}(sx+u)(rx+v) &= ax^2 + mx + nx + c \\ &= ax^2 + (m+n)x + c \\ &= ax^2 + bx + c\end{aligned}$$

This is the original trinomial expression and we are done.

Cathy Falci was originally exposed to the ARCH method during her student-teaching experience at State College High School. She was in the tutoring center and a student asked her if she could show her the method that everyone else in the class was using because they could factor 10 times faster than she could. At the time, Cathy had not heard of this method, but one of the teachers at the high school showed her, and she has been using this method to help students ever since. She finds that it greatly reduces the frustration that many students associate with factoring, especially when the leading coefficient is not prime. This method has been passed down from teacher to teacher as many great teaching ideas are.

We are not advocating this method as a replacement for more formal ways of factoring. However, as a back-up approach, the ARCH method may be a useful alternative for some students who need to see another representation before they master the skill of factoring.

Reference

Pennsylvania Department of Education. (2012). Keystone Exams: Algebra II Assessment Anchors and Eligible Content. Retrieved from http://www.pdesas.org/Standard/CommonCore

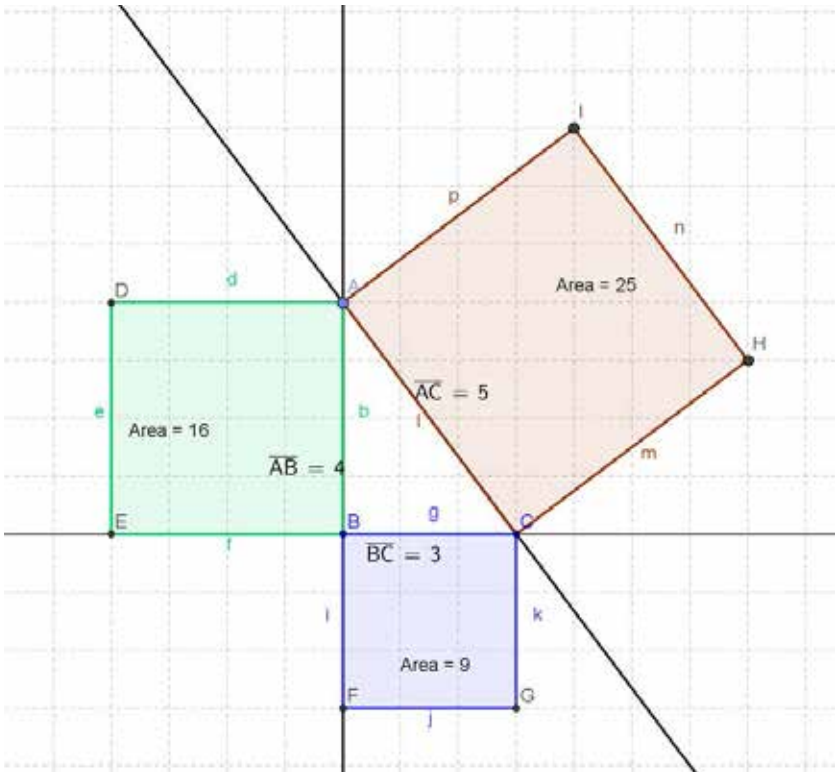
Euclid's "Windmill" Proof of the Pythagorean Theorem

Austin Tolan, Northern Middle School, Northern York County School District

Often, I have found that students can remember the Pythagorean Theorem for the first couple of days or weeks but they have trouble retaining it for the long haul. The reason for this could be that they do not understand what the Pythagorean Theorem actually says. Most people know that it has something to do with finding the side lengths of a right triangle, but if you were to ask them how a shape like a square is involved, they would not know what to tell you. Euclid's "Windmill" proof helps to give us a geometrical representation of how and why the Pythagorean Theorem really works.

Using this "windmill" proof I developed an inquiry-based activity for students to discover the Pythagorean Theorem and be able explain how or why it works. I used the free online software Geogebra to create this activity, and my students really responded well to it. Using the software, students start by creating a right triangle and then constructing squares on each side. Next, they will use the tools in Geogebra to find the lengths of the legs and hypotenuse of the triangle and compare them to the area of each of the squares. Some students are able to make the connection that if we square the length of the leg or hypotenuse then we will get the area of the square, while others need a little push in the right direction. The activity also has the students compare the areas of each of the three squares, and this is where some of them can discover the Pythagorean Theorem. I will usually have my students present their findings to the class. Some students may be a little slower than others using the Geogebra software and I have found it best to pair them up or make groups of three to work through this activity. Working in groups also helps students to talk through and analyze their results.

Geogebra has a nice feature that allows the edges of any shape to "snap" to the grid. This allows the user to be able to work with integers and makes the activity a little easier to navigate through. It also helps in the last part of the activity where I have my students try to find other Pythagorean Triples by make their triangle larger or smaller. Typically, students will have an easy time finding a 3-4-5 and its multiples but they very seldom find others like the 5-12-13. To get students to discover other triples may take a little push in the right direction.



With Geogebra software, a right triangle can be studied in terms of the areas of squares attached to its sides.

Overall, I have found that this activity helps students to get a real conceptual understanding of the Pythagorean Theorem. They have been able to retain the information longer and use it again later in the year.

[See pages 12-13 for the discovery lab sheet that accompanies this article.]

Euclid's "Windmill" Proof of the Pythagorean Theorem

Open Geogebra.

Start by going to View and turn off the axis and turn on the grid.

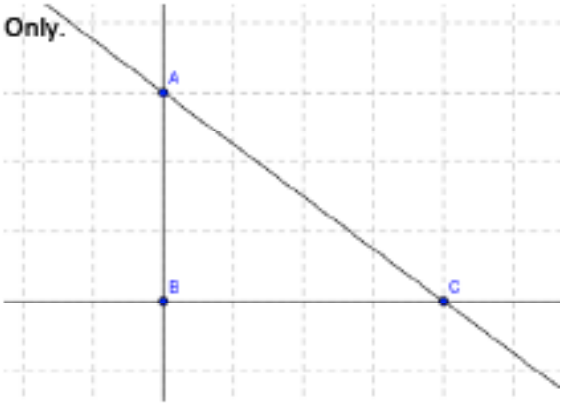
Then go to Options and down to Labeling and select New Points Only.

Construct a vertical line on the worksheet.

Next construct a line perpendicular to the previous line.

Construct segments AB, BC, and AC.

You want your worksheet to look something like this →

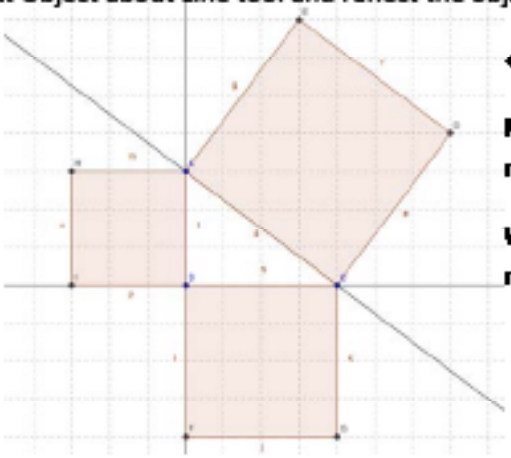


Since the two lines that we made are perpendicular, we know that they meet at 90-degree angles so we now have a right triangle.

Next use the regular polygon tool and choose points A and B and choose 4 for the number of vertices to get a square with side lengths that are the same length as segment AB.

Do this for segments AC and BC also.

If the squares appear and cover the triangle instead of being formed beside the triangle then use the Reflect Object about Line tool and reflect the object across the line that it should be touching.



← We want our picture to look something like this.

Next we can use the Distance or Length tool to get the measures of AB, BC, and AC.

We can also use the Area tool to find the area of our new squares.

$m(\overline{AB}) =$ $m(\overline{BC}) =$ $m(\overline{AC}) =$

Area of Square with side AB = Area of Square with side BC =

Area of Square with side AC =

The area of each square is equal to the measure of the _____ squared.

What do you notice about the relationship between the area of the squares along the legs of the triangle and the area of the square along the hypotenuse of the triangle?

Drag your triangle around a bit. Is the relationship you found always true? Reminder: Make sure that your triangle is always a right triangle.

Now, I would like you to find the area of two squares whose sum is equal to the area of the third but it cannot be the same as the areas that you used above. You may use Geogebra to do this. (Hint: the areas that you use should be perfect squares.) Next, if we take the square root of the areas we will now have the lengths of our legs and the hypotenuse of our right triangle. For Example, the area of the squares made from the legs and hypotenuse of my right triangle are 9, 16, 25 and 9 + 16 = 25. So, the lengths of the sides of my right triangle are 3, 4, and 5. Try to come up with as many examples as you can to fill in the table. These are called Pythagorean Triples. You may find some patterns!

Area of Square A	Area of Square B	Area of Square C	A	B	C
9	16	25	3	4	5

Flatland 2: Sphereland

A movie review by Gail M. Anderson, Christopher Dock Mennonite High School

Many geometry teachers have long been using the 1884 novel *Flatland* by Edwin A. Abbott or the cleverly animated movie version, *Flatland: The Movie* (2007) in their classrooms to help students deepen their understanding of 2-and 3-dimensional geometry in an enjoyable and engaging way. (See my article in *Mathematics Teacher, Vol. 105*, No. 5, December 2011.) More recently released is a sequel to *Flatland: The Movie*, *Flatland 2: Sphereland* (2012). Students and teachers alike will enjoy this exploration into outer "space," non-Euclidean geometry and higher dimensions.

The story of *Flatland: The Movie* ends with the startling discovery of the third dimension by Arthur Square and his granddaughter, Hex. Arthur Square inspires and exhorts students to think and to wonder about what they cannot see: "Mathematics.... reason... imagination... will help reveal the truth." (Arthur Square, *Flatland: The Movie*). *Flatland 2: Sphereland* picks up the story twenty years later. Things have changed in Flatland; Arthur Square has passed on and his granddaughter, Hex, has become a lonely, ridiculed mathematician. Although Flatlanders have progressed into a more egalitarian social structure where all shapes are equal, and have developed a way to begin space exploration, they have failed to accept the existence of "that which they cannot see." The story centers on a new hexagonal mathematician, Puncto, who stumbles upon a surprising set of measurements which promises to shake the very foundations of Flatland: a triangle whose sum of angles exceeds 180 degrees! Puncto is distressed by his data, and, after he can't find anyone to believe him, he seeks out Hex, the only one who may have the imagination and curiosity necessary to help him. The movie follows the adventures of Puncto and Hex as they grapple with this conundrum, fighting against the clock as their fellow Flatlanders embark on an endangered mission to outer space. Aided by their friend from the third dimension, Spherius, and using the helpful and readily-grasped analogies of Pointland and Lineland (familiar to the audience member who has seen *Flatland: The Movie*) they are able to see the curved nature of their own "flat" universe into the third dimension, and to propose the curved nature of Spherius' three-dimensional universe in a fourth dimension, and so on, up the ladder of higher dimensions.

A fun addition to this movie plot is the charming love story between Hex and Puncto; students will enjoy seeing two two-dimensional objects attempt a kiss in 3-D! The movie is inspired by the 1965 novel *Sphereland* by Dionys Burger which is also a great read, but is much more technical. The movie writers take several threads of the book and masterfully tie them together in a modern storyline.

Included with the Educational Version of the DVD are fantastic worksheets on topics of both mathematics and physics. I found the physics worksheet on multiple dimensions very helpful for my own understanding of this abstract concept and would recommend teachers read through that before showing the film, in order to be prepared to answer the difficult questions which are bound to be raised by students following the film. Math worksheets include basic arc length review as well as an excellent foray into spherical geometry.

The movie is only 30 minutes long, which leaves time, even in a single 40-minute class period, to show the *Flatland: The Movie* trailer first (a nice recap if you showed that movie several months ago or if you might have students who hadn't seen *Flatland*) as well to enjoy, or at least start, a lively discussion on higher dimensions. For those with more time or days, the worksheets could be used as final review at the end of the year, as an extension project, or even as a joint physics-math department adventure.

For me, this film presents a perfect ending to a year-long geometry course. In my class, we begin the year with an introduction to Euclidean geometry and I pose the question, "what else is there?" and then we work our way through the classic study of geometry in two and three dimensions. Halfway through the course, we watch *Flatland: The Movie*, which provides an excellent visual in their growing understanding of cross-sections and dimensions, as well as an introduction to our unit on regular polygons. I look forward to bringing the year to a close by suggesting "we have studied Euclidean two and three dimensions quite a bit this year, but there is so much more out there: here's something fascinating to ponder this summer!"

References

Abbott, E. A. (1884). *Flatland: A romance of many dimensions*. Boston, MA: Roberts Brothers.

Burger, D. (1965). *Flatland 2: Sphereland*. New York: Cromwell.

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Flatland 2: Sphereland. (DVD-ROM, 2012) Johnson, D. (Dir.) Flatworld Productions.

Binary Addition, Symbolic Logic, and the Binary Half Adder

Sid Kolpas, Delaware County Community College

In the course *Modern College Mathematics I* at Delaware County Community College, symbolic logic and basic circuit diagrams are taught early in the semester. Toward the end of the semester the binary number system is taught. However, no link is ever made between addition in the binary number system, symbolic logic, and circuitry.

In the binary number system, fundamental to how modern electronic devices compute, addition is done bit-by-bit as follows:

1	(1 plus 1 is 0, carry a 1)	1	(1 plus 0 is 1, carry a 0)
+ 1		+ 0	
10		01	
0	(0 plus 1 is 1, carry a 0)	0	(0 plus 0 is 0, carry a 0)
+ 1		+ 0	
01		00	

To combine the concepts of symbolic logic, circuitry, and binary addition, I introduce the Binary Half Adder circuit, which adds two binary digits (bits). My students have already mastered truth tables for

"and" ($\wedge$), " or" ($\vee$), "not" (~), conditionals ($p \rightarrow q$),

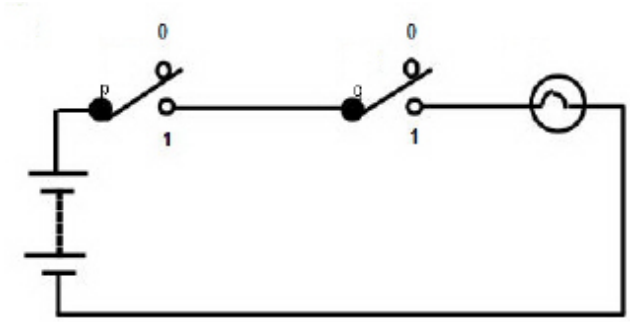
and biconditionals ($p \leftrightarrow q$). They also know that

$p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

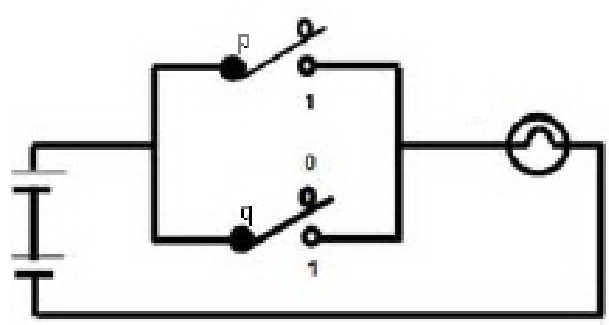
and that $p \rightarrow q$ is equivalent to $\sim p \vee q$.

Additionally, they know DeMorgan's Laws.

p and q represent bits where 1 = true and 0 = false. Students have also learned that "or" can be represented by a parallel circuit and "and" by a series circuit.



An AND circuit



An OR circuit

The sum of two binary digits is 0 if the bits are the same and 1 if the bits are different. Students are quick to note that for the biconditional, a statement is true (= 1) if both p and q are the same value, and false (= 0) if p and q are different values. We want just the opposite of this for the sum, so we need to negate the biconditional.

$\sim(p \leftrightarrow q) \equiv$

$\sim((p \rightarrow q) \wedge (q \rightarrow p))$ by the definition of the biconditional $\equiv$

$\sim((\sim p \vee q) \wedge (\sim q \vee p))$ substituting equivalent expressions $\equiv$

$\sim(\sim p \vee q) \wedge \sim(\sim q \vee p)$ by Demorgan's Laws $\equiv$

$(p \wedge \sim q) \vee (\sim p \wedge q)$ by Demorgan's Laws

So, a symbolic expression for the ***sum*** is

$$(p \wedge \sim q) \vee (\sim p \wedge q)$$

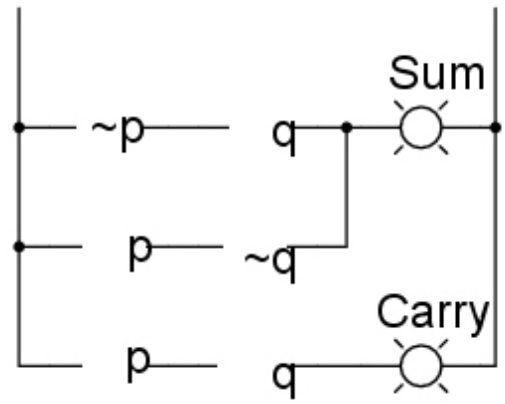
The carry of two binary digits p and q is 0 if either p or q is 0, and 1 if they are both 1. But this matches the truth table for $p \wedge q$

So, a symbolic expression for the ***carry*** is

$$p \wedge q$$

Students fill out a truth table to verify these expressions work for adding 2 bits:

Finally, students are asked to diagram a Binary Half Adder Circuit, which adds two bits.



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						SUM	CARRY
p	q	~p	~ q	(p ^ ~q)	(~p ^q)	(p ^ ~q) v (~p ^ q)	(p ^ q)
1	1	0	0	0	0	0	1
1	0	0	1	1	0	1	0
0	1	1	0	0	1	1	0
0	0	1	1	0	0	0	0

Incorporating Topics from the History of Mathematics into the Teaching of Trigonometry: Exploring Applications to Economic Cycles

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In this article, we summarize a successful approach for exploring the historical origins of trigonometry by illustrating the possibility of plotting sums of trigonometric functions in pre-calculus courses. This exercise is preceded by a discussion of the origins of trigonometry in ancient Babylonia, Greece, and Egypt, and of how trigonometry was used by Islamic astronomers. We believe that coverage of such topics can nurture students' curiosity regarding cyclical phenomena such as national economies.

Plotting Trigonometric Functions

In pre-calculus courses, coverage of graphs of standard trigonometric functions has been enhanced by the exercise described below.

Students have been directed to examine altering the amplitudes and frequencies of sinusoidal functions by plotting $f(x) = \sin(x)$, $g(x) = \frac{1}{3}\sin(3x)$, and $h(x) = \frac{1}{5}\sin(5x)$ over the interval $-4\pi \le x \le 4\pi$. Most students have had little difficulty in producing by hand graphs similar to the ones shown in figures 1, 2 and 3.

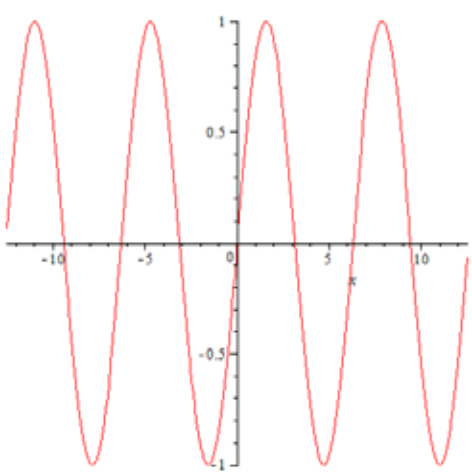

Figure 1: graph of $f(x) = \sin(x)$

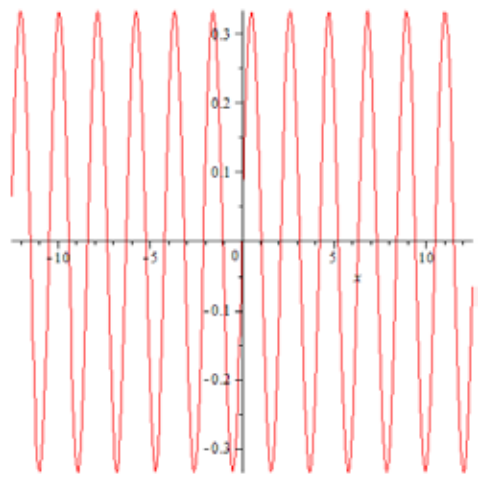

Figure 2: graph of $g(x) = \frac{1}{3}\sin(3x)$

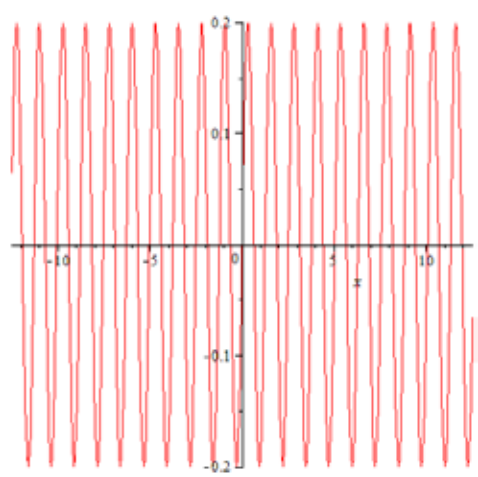

Figure 3: graph of $h(x) = \frac{1}{5}\sin(5x)$

In what initially appears to be a completely unmotivated exercise, students are asked to graph the sum of the first two functions $\sin(x) + \frac{1}{3}\sin(3x)$ using graphing calculators. Most students are able to produce a graph similar to the one shown in figure 4,

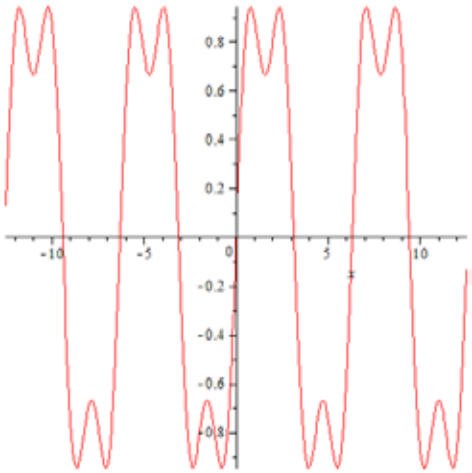

Figure 4: graph of $f(x)+g(x)$

although some students require some assistance in typing in the correct code. Several students have spontaneously asked, "What would happen if we graphed the sum of all three functions?" and have graphed the sum:

$$\sin(x)+\tfrac{1}{3}\sin(3x)+\tfrac{1}{5}\sin(5x),$$

obtaining the graph shown in figure 5:

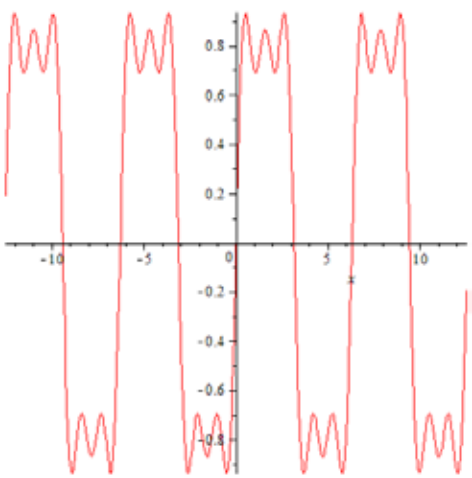

Figure 5: graph of $f(x)+g(x)+h(x)$

At this stage, the curiosity shown by the students who asked the prior question became contagious, and the class has enthusiastically volunteered:

Let's keep going !

The students then attempted to graph sums of the form:

$$\sin(x)+\tfrac{1}{3}\sin(3x)+\tfrac{1}{5}\sin(5x)+\tfrac{1}{7}\sin(7x)+\tfrac{1}{9}\sin(9x)+\tfrac{1}{11}\sin(11x)+\tfrac{1}{13}\sin(13x)+\ldots \qquad (1)$$

often seeking assistance from each other or from the instructor in determining the terms in the sum, but eventually generating the graph shown in figure 6:

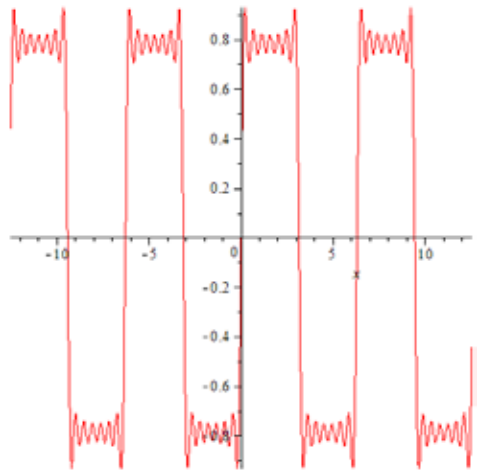

Figure 6: graph of trigonometric sum

Toward the end of class, students are asked,

If you could keep going forever, what would the graph look like?

After a few minutes of reflection, several students volunteer to display their conjecture at the board, producing the graph shown in figure 7.

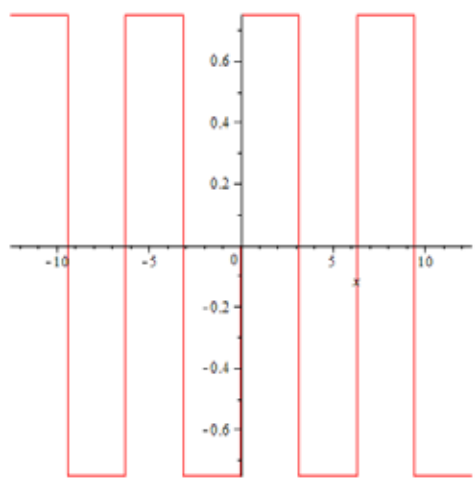

Figure 7: conjectured limiting graph

Playing devil's advocate, the instructor has shaded in the sharp spike that occurs near multiples of π in figure 6 and asked,

Even here?

The class concludes with a homework assignment in which students are directed to use a graphing calculator to plot the sum:

$$\sin(x)+\tfrac{1}{2}\sin(2x)+\tfrac{1}{4}\sin(4x)+\tfrac{1}{6}\sin(6x)+ \tfrac{1}{10}\sin(10x)+\tfrac{1}{12}\sin(12x)+\tfrac{1}{14}\sin(14x)+\ldots \quad (2)$$

and to write a brief biographical report on Jean Baptiste Joseph Fourier.

During the next class meeting, upon comparing notes, students concur in that the sum in (2) produces a sawtooth-shaped repetitive pattern and that Fourier is credited with having been the first to use sums such as (1) and (2) to model processes which are periodic or repetitive. Some students have asked why Fourier was ever motivated to study such sums in the first place, and we have responded that these came up "accidentally" in solving a problem related to the distribution of temperature in a metal rod.

The class is informed that the spike alluded to the previous day is referred to as the "Gibbs phenomenon," and is named after Josiah Willard Gibbs, one of the first American scientists to win international recognition.

This exercise in plotting trigonometric sums has provided pre-calculus students glimpses of the road ahead on their mathematical journeys, shedding insight into the critical role played by elementary trigonometric functions in describing phenomena which are cyclical or have cyclical components.

Modeling Economic Cycles

Many teachers will have students whose families were directly affected by the recent recession. Pointing out that the slowdown from which the U.S. appears to be emerging was preceded by periods of prosperity as well as by various crises, including particularly severe ones in the 1870s and 1930s, may help to assure students that better economic times await them. Economists and economic historians have kept records of business cycles that occurred in the U.S., Germany, and czarist Russia since early in the 19th century. These studies document a pattern of recessions and depressions which have occurred on a fairly regular basis to the extent that the relative economic health of the U.S. has resembled the square wave drawn in figure 7, even though the duration and intensity of recessions has varied.

When modeling economic time-series, forecasters often attempt to model oscillatory economic data with series of the form:

$$Y_i = \sum_j \left[X_j \cos(2p\, f_j t) + Z_j \sin(2p\, f_j t) \right] + m \quad (3)$$

where Y_i is the time series set, X's and Z's are uncorrelated random variables with zero mean and variances denoted by $\sigma^2(f_j)$. Although a thorough development of this topic is well beyond the scope of any high school mathematics course, pointing out that representations such as (3) are generalizations of sums such as (2) and (3) will lead students to speculate about the extent to which a nation's economy is cyclical or periodic in nature, and alert them that sines and cosines are essential building blocks in describing such periodicities. Such explorations will allow for greater integration of trigonometry into the overall academic curriculum and to a heightened awareness of the central role played by mathematics in economics and other social sciences.

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