# TABLE OF CONTENTS

## PREFACE

v  
Matthew DeGraaf, Teachers College Columbia University  
Simone Salmon-Nembhard, Teachers College Columbia University

## ARTICLES

1  
**Rainforest Mathematics**  
Jeremy Kilpatrick, University of Georgia

9  
**Using a Framework of 21st Century Competencies to Examine Changes between China’s 2001 and 2011 Mathematics Curriculum Standards for Basic Education**  
Max Stephens, University of Melbourne, Australia  
Richard Xu Keqiang, South West University, China

17  
**Solving Optimization Problems with Dynamic Geometry Software: The Airport Problem**  
José Contreras, Ball State University

29  
**Using Dynamic Software to Address Common College Calculus Stumbling Blocks**  
Alice W. Seneres, Rutgers, The State University of New Jersey  
John A. Kerrigan, Rutgers, The State University of New Jersey

39  
**Rousing Students’ Minds in Postsecondary Mathematics: The Undergraduate Learning Assistant Model**  
David C. Webb, University of Colorado Boulder  
Eric Stade, University of Colorado Boulder  
Ryan Grover, University of Colorado Boulder

49  
**Using Mathematics Literature with Prospective Secondary Mathematics Teachers**  
Christopher C. Jett, University of West Georgia

55  
**Financial Literacy: An Essential Component of Mathematics Literacy and Numeracy**  
Marla A. Sole, Guttman Community College, The City University of New York

63  
**Integrating Universal Design and Response to Intervention in Methods Courses for General Education Mathematics Teachers**  
Kelley Buchheister, University of South Carolina  
Christa Jackson, Iowa State University  
Cynthia E. Taylor, Millersville University of Pennsylvania
INTEGRATING UNIVERSAL DESIGN AND RESPONSE TO INTERVENTION IN METHODS COURSES FOR GENERAL EDUCATION MATHEMATICS TEACHERS

Kelley Buchheister  
University of South Carolina

Christa Jackson  
Iowa State University

Cynthia E. Taylor  
Millersville University of Pennsylvania

ABSTRACT Traditionally, teacher education programs have placed little emphasis on preparing mathematics teachers to work with students who struggle in mathematics. Therefore, it is crucial that mathematics teacher educators explicitly prepare prospective teachers to instruct students who struggle with mathematics by providing strategies and practices that specifically address their needs. In this study, we describe the principles of Universal Design for Learning and Response to Intervention. More specifically, we discuss how one Mathematics Teacher Educator uses these frameworks in her mathematics methods course to help prospective teachers become cognizant of early interventions and effective strategies that can be implemented to provide all students with the greatest opportunity to learn.

KEYWORDS equity, Universal Design for Learning, Response to Intervention, prospective teachers, teacher education, mathematics teacher educator

The National Council of Teachers of Mathematics (NCTM), *Principles to Actions: Ensuring Mathematical Success for All* (2014), describes a strong mathematics program as one that exemplifies reasoning and sense-making, which is achieved through authentic discussions, experiences, and tasks. Moreover, the authors argue that an effective mathematics program requires all students have the opportunity to engage in mathematical tasks that require a high level of cognitive demand. Implementing this vision can be challenging; yet, even more demanding is how to support students who struggle with mathematics (van Garderen, Scheuermann, Jackson, & Hampton, 2009).

Traditionally, teacher education programs have placed little emphasis on preparing mathematics teachers to work with struggling learners (Allsopp, Kyger, & Lovin, 2007). However, the likelihood that a prospective elementary, general education teacher will have a student who experiences mathematical difficulties in his or her classroom is high. In fact, approximately “5 – 10% of elementary general education students have difficulties with mathematics” (Kroesbergen & Van Luit, 2003, p. 97). Although the general education classroom includes more and more students with varying levels of content knowledge and experience, the reality is that there is little attention to designing lessons or classroom environments that “meet the needs of a more diverse population of students possessing broader ranges of skills” (Mackey, 2012, p. 1). Recent surveys demonstrate that general education teachers who are responsible for Pre-K through 12th grade mathematics instruction have limited knowledge of specialized practices that can assist students with disabilities or who struggle with mathematics (Macinni & Gagnon, 2002, 2006). Specifically, students who have struggled are more likely to have teachers who have weaker math backgrounds, less professional experience, and certification outside of rather than in mathematics and who are perceived to
be less effective...[and provide] lessons that focus primarily on rote skills and procedures with scant attention to meaningful mathematics learning. (NCTM, 2014, p. 61)

Effective equitable instruction is achieved when the needs of all students in the classroom are supported (NCTM, 2014; Smith & Tyler, 2011), including those who experience difficulties with mathematics. Equity does not imply identical instruction, but focuses on instruction that includes appropriate accommodations, which provide opportunities for students to learn and be engaged in rigorous mathematics (NCTM, 2000). Therefore, it is crucial that we, as mathematics teacher educators (MTEs), explicitly prepare prospective teachers (PTs) to instruct students who struggle with mathematics by providing strategies and practices that specifically address their needs.

In this article, we explain the principles of Universal Design for Learning (UDL) and Response to Intervention (RtI), which can promote students’ active participation and encourage both students and teachers to present information in a variety of ways (McNulty & Gloeckler, 2011; Rose & Meyer, 2000, 2002). We describe how one mathematics teacher educator integrates the principles of UDL and RtI as a lens through which prospective teachers can analyze, modify, and apply pedagogical strategies focused on developing the mathematical knowledge and skills of all students.

Frameworks for Equitable Practice

Universal Design for Learning

“To provide access and equity, teachers go beyond ‘good teaching,’” to teaching that ensures that all students have opportunities to engage successfully in the mathematics classroom and learn challenging mathematics” (NCTM, 2014, p. 68). In concert with this idea lie the principles of UDL. With the knowledge that retrofitted accommodations are often not sufficient, the notion of UDL originated in architecture as a means to create structures that attended to the needs of not only those with disabilities, but also those from a diverse population (Dolan & Hall, 2001). The theory was applied to education with rising concerns about how special education students mainstreamed into the general education classroom would gain access to the general education curriculum and standards (Edyburn, 2010). It is important to recognize that this theory is relevant to all students, not just students in special education. By anticipating students’ potential barriers and obstacles, classroom teachers can provide opportunities for all students to achieve success in mathematics.

In educational settings, UDL alters the emphasis from considering the child as “deficient,” and instead focuses on what the instructor can do to change practice or the classroom environment in order to support the needs of the learners. It is rooted in the philosophy that if one anticipates individuals’ needs and considers strategies to accommodate those needs from the outset, then there are unforeseen benefits for all individuals. Through this design (see Figure 1), teachers consider flexible methods of presentation (e.g., multiple modes of representation, varied contexts or situations), expression (e.g., share mathematical thinking through various modalities and mediums), and engagement (e.g., being aware of learners’ interests and strengths) (Basham & Marino, 2013).

Response to Intervention

The RtI framework is being adopted within more and more districts in order to increase students’ mathematical success in the general education classroom (Riccomini & Witzel, 2010). The RtI framework is generally based on a three-tiered system (see Figure 2), which was designed to structure support and provide...
appropriate interventions to reduce the need for special education services.

Approximately 80% of students fall within Tier 1 and these children work toward the general education curriculum expectations (Riccomini & Witzel, 2010). However, even amongst students in Tier 1 there is a need to differentiate instruction and provide additional support. For those students who do not respond to the implemented modifications, the classroom teacher transitions these students to receive Tier 2 instructional supports in small group instruction. Tier 2 supports are systematic, focusing on process and skill development for students who experience difficulty in mathematics. For example, novice learners—and students who struggle with mathematics—often attend to more superficial details rather than the relevant information or conceptual relationships embedded in mathematical problems. Students who experience this difficulty often represent the problem through pictorial images that primarily depict non-mathematical features, such as the visual appearance of the objects or people described in the given problem, instead of schematic images that depict relationships among the quantities in the problem (van Garderen, 2007). Therefore, a Tier 2 support could include providing empirically-based, focused instruction that facilitates the development of schematic diagrams, such as using tape diagrams to represent the underlying structure of the problem (See Figure 3).

**Problem:** Lane had 47 pennies. Then she found some more pennies. Now Lane has 63 pennies. How many pennies did Lane find?

<table>
<thead>
<tr>
<th>47</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>63</td>
</tr>
</tbody>
</table>

**Figure 3.** Change problem represented with tape diagram. (adapted from Gersten Beckmann, Clarke, Foegen, Marsh, Star, & Witzel, 2009)

Representations, like the tape diagram in Figure 3, can aid students in identifying common underlying structures of mathematical problems—particularly in word problems or contextual situations—and assist them as they distinguish “substantive information from superficial information, in order to solve problems that fit into a category of problems that they already know how to solve” (Gersten, Beckman, Clarke, Foegen, Marsh, Star, & Witzel, 2009, p. 27). Tier 2 instructional supports, like teaching students strategies to identify underlying structure of various problem types, are more explicit and empirically-based interventions that give students additional instructional time—approximately 25 minutes—to help them focus on specific content within small group settings.

Although instruction for both Tier 1 and Tier 2 occur within the general education classroom, Tier 3 interventions may occur outside of the general education setting. Tier 3 includes intense interventions and is designed for about 5% of the students in the classroom who are unresponsive to evidence-based interventions in Tiers 1 and 2. These students receive a higher intensity mathematics program with different core concepts and skills from their Tier 1 and 2 peers and also undergo diagnostic assessments (e.g., Math Recovery, Key Math) to identify appropriate interventions and services.
Interweaving UDL and RtI
Based on the principles of UDL and RtI, it is clear the elements of the two frameworks can co-exist within the general education classroom and be implemented in order to attend to the needs of a diverse population of students (see Figure 4).

Figure 4. Combining key elements of UDL and RtI (adapted from Dalton & Ambruzzini, 2010)

As PTs begin to recognize the key principles of UDL, MTEs can discuss how identifying potential barriers and implementing instructional strategies that address these obstacles may be implemented in Tier 1 of the RtI framework. For example, using data from formative and summative assessments to examine curriculum materials in the context of students’ strengths, interests, and needs provides teachers with opportunities to determine what supports would be necessary in order to help students in the first tier positively respond to the mathematical lesson. Then more explicit interventions and supports (e.g., schema-based instruction, ixl tutorials) may be provided during small group work to aid students in Tier 2 who do not respond to the initial adaptations to the lesson.

The Lesson

In the next section, we describe the story of implementing UDL and RtI within an Early Childhood (Pre-K through Grade 3) Mathematics Education Course for general education PTs taken the semester before their student teaching internship. The participants in the Spring 2013 semester included one section of the undergraduate course for Early Childhood Education majors (22 females and 1 male). Using the “Boa Constrictor” problem as a focus of instruction, the MTE emphasized algebraic thinking and reasoning in the context of the 7th Standard of Mathematical Practice (SMP), “look closely to discern a pattern or structure” (The National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 8). As PTs participated in and analyzed the lesson, they were provided a practical experience in which they applied the principles of UDL and RtI to explore curriculum and instructional strategies through an equity lens.

The Boa Constrictor

The lesson, Boa Constrictor, was based on Shel Silverstein’s poem and was structured to model the Launch-Explore-Summarize sequence, which is a common lesson format in K-12 mathematics curriculum. In the lesson, students were expected to engage in the thinking corresponding to SMP 7 by examining, describing, building, and extending a growing pattern for a fictional boa constrictor. At the beginning of the lesson, the class discussed what they knew about the boa constrictor, and conversations quickly focused on the “biggest” snakes they had ever seen. The MTE asked the PTs to clarify what they meant by the “biggest,” and they came to a consensus that length was the correct mathematical term.

Next, the MTE introduced the PTs to the following exploration problem for third graders:

Riverbanks Zoo has a new boa constrictor to add to the Aquarium Reptile Complex. Currently, the length of the snake is two segments with the measurement beginning at the base of its head. The snake is expected to grow three segments each week. If the snake continues to grow at this rate, how many segments long will the snake be at the end of the month?

The PTs were presented with a variety of mathematical tools (e.g., pattern blocks, markers, construction paper, multi-link cubes), as well as a calendar of the current month, to solve the problem. As the MTE circled the room, she asked the PTs about difficulties they experienced with the written task or in representing the snake and the strategies they used to determine key information needed to solve the problem. In addition, the MTE encouraged PTs to describe any patterns or relationships they noticed. Through these guiding questions, the MTE purposefully drew PTs attention toward potential barriers their elementary students might
experience as they participate in the lesson. For the PTs who finished early, the MTE asked them to find the number of segments in a snake that grew at that rate for 210 days. This extension problem encouraged students to use the pattern to identify an explicit rule rather than continue with the recursive rule, which can be cumbersome at this point in the sequence.

After the PTs created a representation of the problem and attempted the extension to the boa constrictor task, the MTE selected three different solutions (see Figure 5) to analyze in a large group discussion. The MTE was strategic in her selections in order to emphasize various modes of representation and presentation of the task. Offering variety in presentation is one of the key elements in the UDL framework. Embedding this element in the classroom discussions reminded the PTs of the value in having students compare and contrast the strategies as they discussed: (a) how each strategy corresponded to the task, and (b) possible errors or alternative interpretations.

During the discussion, the PTs first discussed how they needed to focus on the “full weeks” in the current month. Most PTs eliminated the additional days (at the beginning and end of the month that were not full weeks), however, upon reflection several PTs recognized that they could have combined the days at the beginning and end of the month to form another full week. In fact, Deanna noted “we could put these days together with these days and that would make another week, which would make this more accurate.” Furthermore, by strategically selecting and presenting three different representations of the problem (Figure 5) the MTE provided the PTs with an opportunity to consider how their solutions would differ depending on how they interpreted “weeks” when provided with a monthly calendar.

The representations the MTE selected also encouraged the PTs to identify how each showed the growth of three segments per week. For instance, they recognized the growth pattern through the sets of three shapes in Snake A, the equation “3×number of weeks” in Snake B, and in the table as an increase in y in Snake C (Figure 5). While the PTs successfully identified an error in the table—there were only two segments in Week 0 and not three as indicated—the students struggled with the calculation error in the equation. In fact, Dianna commented,

Something is not right. The equation is right because it’s two to start and then you add three each week, so that’s three times the week. Plus the two. And that’s also in the blocks right here. But, the math is right because 2 + 3 is five and then 5 × 4 is twenty.

Following much confusion and debate, one student was able to clarify the misconception as he closely examined the different presentations of the growth pattern. D’Travis explained,

The three times part is just for the weeks. If you did 3 + 2 first and then multiplied that by the weeks that’s not right. The snake is not growing 5 segments each week. It’s only growing three. So the two is extra and has to be added on after you do the multiplication. You should write it 3 × 4 + 2—wait! You don’t have to—you can just put parentheses around it and do that part first. So then you’d get 14 because 3 × 4 is 12 and then plus two more you get 14. That’s the same.

Although creating an explicit formula was an obstacle for many PTs, they were more successful after the discussion of the presented formula (2 + 3 × week). Moreover, several PTs referenced SMP 7 describing how the extension task “forced” them to investigate patterns within their tables and physical representations.

After completing the lesson activities, the MTE asked the PTs to consider the lesson through a UDL lens, specifically reflecting on their personal strengths and challenges as well as the identified areas of difficulty for students who struggle with mathematics. The MTE emphasized that by acknowledging and addressing these areas in the early stages of planning the PTs were providing students with a greater opportunity to learn. For example, scaffolds can be inserted into the general large group lesson in order to directly meet the needs of the Tier I students, while the PTs can also consider the more challenging needs of students in their classroom and implement specific evidence-based interventions.
that will provide additional support to students in Tier II. Incorporating the UDL lens provides PTs with the opportunity to identify obstacles elementary students could face and potential strategies to accommodate those areas.

Most of the PTs claimed there was an appropriate level of complexity for third grade students in the activity, but they argued that the discussion could be more focused to make connections between the task, the table, and the algebraic pattern because it could be challenging for students who struggle with mathematics to translate across multiple representations. The PTs emphasized the teacher should continually go back and forth across each representation and discuss the similarities and mathematical relationships among the representations. For example, Brittany suggested it was important for students to share their solutions and representations, but she argued the discussion should include a whole class discussion in which the students transfer the data in the original problem to the physical representation, the table, and the equation—and then make connections across the multiple modes of representation. Brittany claimed this process would help students who were struggling to see the relationship across the representations and prompt students as they looked for similarities and differences among the solution strategies. Furthermore, as a group, the PTs stated that more direct instruction would provide a scaffold for primary students—especially those with mathematical difficulties—and help them generalize the relationships in the growing pattern.

Although several PTs noted the launch was “cute,” they argued it did not explicitly transition students to work with patterns. One group suggested that after the poem students could engage in patterning tasks where they repeat patterns, extend patterns, fill in the blanks, or create patterns before they are asked to find the solution to the task. Rachael’s group explained it would be beneficial to discuss the problem and clarify what Week 0 and Week 1 looked like before letting the students “go off on their own.” However, not all PTs agreed with this suggestion. For example, another group argued that the lack of clarity led to discrepancies in the data during the summary portion of the lesson, which stimulated valuable discussions and deeper thinking about key information in the problem. From this conversation, the group decided it was essential to clarify elements of the initial problem (e.g., head vs. segment), but modeling the weeks should serve as an additional prompt for struggling students.

The MTE explained each suggested modification could help students in the first tier gain access to the problem, but also emphasized they would need to consider explicit interventions for students who did not respond to initial adaptations. The class discussed how guided practice during mathematics centers could provide an opportunity to focus small groups of students on specific obstacles related to mathematical content or processes. For example, the MTE demonstrated the “copy, cover, compare” method, often used as an intervention for computation practice, as a Tier 2 support for patterning tasks. The PTs analyzed how to incorporate this strategy in guided practice: view a pattern, copy the pattern, cover the example, and then repeat the pattern and compare the results. This process would provide a scaffold for students as they examined patterns.

**Lesson Modification**

After the discussion of the Boa Constrictor task, the MTE provided the PTs with one lesson from a textbook commonly used in their district. This activity did not emphasize growing patterns as in the previous task, but had the learners investigate patterns and identify rules in in/out tables (Figure 6).

<table>
<thead>
<tr>
<th>In</th>
<th>Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>7</td>
</tr>
<tr>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 6.** “What’s My Rule” in/out table.

The PTs were asked to use the principles of UDL and RtI to anticipate barriers in the mathematical task and use their methods text as a guide to design appropriate modifications that would address these potential obstacles in the first tier. Then the PTs were asked to design centers that could be used for Tier 1 and Tier 2 supports. PTs considered additional activities they could use to target specific needs of students: (a) practice established pattern skills, (b) extend the concept to novel situations, and (c) provide a focused, small group intervention to accommodate students in Tier 2. For instance, a group of PTs selected tasks that provided opportunities for children to build patterns with materials like buttons, stickers, blocks, or cubes to reinforce pattern structures. The PTs explained that children at this station could either choose to extend patterns displayed on
cards already at the table, or they could choose to create their own pattern and represent the structure with a corresponding letter sequence such as ABBA. With either choice, all elementary students, with the exception of those who might need additional modification, would be expected to record the pattern in their mathematics journal by drawing the pattern using crayons, pencils, or a stencil. In other examples, the PTs designed differentiated activities where children explored patterns by: (a) creating or listening to patterns in music or dance, (b) looking for patterns in nature pictures, (c) reading poetry to find rhyming patterns, or (d) completing pattern activities on web-based activities like the function machine on the “math playground.” Finally, students applied interventions from Papic’s (2007) article, Promoting repeating patterns with young children—More than just alternating colors. Several PTs incorporated tasks from the article (i.e., linear (tower) patterns, cyclic (border) patterns, and hopscotch patterns) as Tier 2 supports in a guided practice station.

After the PTs had the opportunity to develop their lessons on patterns, the group displayed their ideas and participated in a gallery walk. Following the gallery walk, the class discussed the development of guiding questions and prompts, asked clarifying questions regarding the Tier 1 or 2 interventions, highlighted effective transitions across the lesson sequence, and reiterated the big ideas within the conjoined UDL and RtI framework. This conversation emphasized effective pedagogical practices designed to meet the needs of students who struggle with mathematics.

Summary

One factor contributing to students’ difficulties with mathematics is the contrast between the needs of an individual student and the type of instruction she or he receives (Kroesbergen & Van Luit, 2003). In order to overcome some of the obstacles students who struggle with mathematics face, PTs need to become cognizant of and gain experience with early interventions and effective strategies that may be implemented to provide all students with the greatest opportunity to learn. One support for PTs includes providing an explicit framework for curriculum analysis, lesson planning, and implementing instructional strategies based on the principles of UDL and RtI. This framework provides a specific lens through which PTs can deeply consider potential barriers that restrict equitable access to the mathematics in the lesson, and identify strategies to “support student engagement by presenting information in multiple ways, and allowing for students to access and express what they know in a variety of ways, [while also including] accommodations that should not alter the standards nor lower the expectations for students” (McNulty & Gloeckler, 2011, p. 6). Mathematics teacher educators who incorporate the UDL and RTI frameworks in their methods courses provide valuable opportunities for PTs to apply their knowledge of student learning progressions and effective instructional strategies, while simultaneously demonstrating how these pedagogical practices are effective tools for engaging a diverse population of students in high quality mathematics.

Although many PTs are familiar with the phrase “differentiating instruction,” the reality of providing differentiated supports and appropriate accommodations to students experiencing difficulties in mathematics is complex (Andreasen & Hunt, 2012). “The question is not whether all students can succeed in math but whether the adults organizing math learning opportunities can alter traditional beliefs and practices to promote success for all” (NCTM, 2014, p. 61). Consequently, it is imperative PTs have experiences in their teacher preparation programs that provide opportunities to explore empirically-based strategies and practices that attend to the needs of a heterogeneous population of students. The activities discussed in this article focus on helping PTs understand two critical elements in teaching students who struggle with mathematics: (a) modeling the principles of UDL and (b) using the RtI model and mathematics centers as a framework to differentiate and provide explicit, empirically-based instruction. By incorporating these frameworks as a lens through which PTs pro-actively analyze mathematics teaching in a methods course, MTEs can emphasize the importance of engaging all students in high quality experiences from the beginning rather than considering adaptations or modifications to lessons as an afterthought.
References


