

**ESCI 341 – Atmospheric Thermodynamics**  
**Lesson 1 – Math Review**

**Partial Derivatives and Differentials**

- The differential of a function of two variables,  $f(x, y)$ , is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad (1)$$

- Eq. (1) is true regardless of whether  $x$  and  $y$  are independent, or if they are both composite functions depending on a third variable, such as  $t$ .
- The terms like  $\partial f/\partial x$  and  $\partial f/\partial y$  are called partial derivatives, because they are taken assuming that all other variables besides that in the denominator are constant.
  - For example,  $\partial f/\partial x$  describes how  $f$  changes as  $x$  changes (holding  $y$  constant), and  $\partial f/\partial y$  describes how  $f$  changes as  $y$  changes (holding  $x$  constant).
- If  $f$  is a function of three variables,  $x$ ,  $y$ , and  $z$ , then the differential of  $f$  is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz. \quad (2)$$

- We often write the partial derivatives with subscripts indicating which variables are held constant,

$$df = \left(\frac{\partial f}{\partial x}\right)_{y,z} dx + \left(\frac{\partial f}{\partial y}\right)_{x,z} dy + \left(\frac{\partial f}{\partial z}\right)_{x,y} dz,$$

though it is not absolutely necessary to do so.

- That partial and full derivatives are different can be illustrated by dividing Eq. (1) by the differential of  $x$  to get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \quad (3)$$

- From Eq. (3) we see that the full derivative and the partial derivative are equivalent only if  $dy/dx$  is zero, or if the partials of  $f$  with respect to the other variables are zero.
- **WARNING!** Partial derivatives are not like fractions. The numerators and denominators cannot be pulled apart or separated arbitrarily. Partial derivatives must be treated as a complete entity. So, you should **NEVER** pull them apart as shown below

$$\frac{\partial f}{\partial t} = ax^2 \Rightarrow df = ax^2 \partial t. \text{ **NEVER DO THIS!**}$$

With a full derivative this is permissible, because it is composed of the ratio of two differentials which can each stand on their own. But there is no such thing as a *partial differential*,  $\partial f$ .

### THE CHAIN RULE

- If  $x$  and  $y$  are not independent, but depend on a third variable such as  $s$  [i.e.,  $x(s)$  and  $y(s)$ ], then the chain rule is

$$\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds}. \quad (4)$$

- If  $x$  and  $y$  depend on multiple variables such as  $s$  and  $t$  [i.e.,  $x(s,t)$  and  $y(s,t)$ ], then the chain rule is

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \end{aligned} \quad (5)$$

### THE PRODUCT RULE AND THE QUOTIENT RULE

- The product and quotient rules also apply to partial derivatives:
  - The *product rule*

$$\frac{\partial}{\partial x}(uv) \equiv u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x}. \quad (6)$$

- The *quotient rule*

$$\frac{\partial}{\partial x} \left( \frac{u}{v} \right) \equiv \frac{1}{v^2} \left( v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right). \quad (7)$$

### PARTIAL DIFFERENTIATION IS COMMUTATIVE

- Another important property of partial derivatives is that it doesn't matter in which order you take them. In other words

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \equiv \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial^2 f}{\partial y \partial x}.$$

- Multiple partial derivatives taken with respect to different variables are known as *mixed* partial derivative.

## OTHER IMPORTANT IDENTITIES

- The reciprocals of partial derivatives are:

$$\left(\frac{\partial f}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial f}\right)_y} ; \quad \left(\frac{\partial f}{\partial y}\right)_x = \frac{1}{\left(\frac{\partial y}{\partial f}\right)_x}$$

- If a function of two variables is constant, such as  $f(x, y) = c$ , then its differential is equal to zero,

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0. \quad (8)$$

- In this case,  $x$  and  $y$  must be dependent on each other, because in order for  $f$  to be a constant, as  $x$  change  $y$  must also change. For example, think of the function

$$f(x, y) = x^2 + y = c. \quad (9)$$

- Eq. (8) can be rearranged to

$$\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dy} + \left(\frac{\partial f}{\partial y}\right)_x = 0. \quad (10)$$

The derivative  $dx/dy$  in Eq. (10) is actually a partial derivative with  $f$  held constant, so we can write

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_f + \left(\frac{\partial f}{\partial y}\right)_x = 0,$$

which when rearranged leads to the identity

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial y}{\partial f}\right)_x \left(\frac{\partial x}{\partial y}\right)_f = -1. \quad (11)$$

- Eq. (11) is only true if the function  $f$  is constant, so that  $df = 0$ .

## INTEGRATION OF PARTIAL DERIVATIVES

- Integration is the opposite or inverse operation of differentiation.

$$\begin{aligned}\int_a^b \frac{\partial f(s,t)}{\partial s} ds &= f(b,t) - f(a,t) \\ \int_a^b \frac{\partial f(s,t)}{\partial t} dt &= f(s,b) - f(s,a)\end{aligned}\tag{12}$$

## DIFFERENTIATING AN INTEGRAL

- If an integration with respect to one variable is then differentiated with respect to a separate variable, such as

$$\frac{\partial}{\partial t} \int_a^b f(s,t,u) ds$$

the result depends on whether or not the limits of integration,  $a$  and  $b$ , depend on  $t$ .

- In general, if both  $a$  and  $b$ , depend on  $t$ , the result is

$$\frac{\partial}{\partial t} \int_{a(t,u)}^{b(t,u)} f(s,t,u) ds = \int_{a(t,u)}^{b(t,u)} \frac{\partial f(s,t,u)}{\partial t} ds + f(b,t,u) \frac{\partial b}{\partial t} - f(a,t,u) \frac{\partial a}{\partial t}.\tag{13}$$

- If  $a$  does not depend on  $t$  then the term in Eq. (13) that involves  $\partial a/\partial t$  will disappear. Likewise, if  $b$  does not depend on  $t$ , then the term containing  $\partial b/\partial t$  will be zero.