ESCI 343 – Atmospheric Dynamics II
Lesson 4 – Introduction to Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Waves in Fluids, J. Lighthill
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.2

GENERAL

The governing equations support many wavelike motions (waves are broadly defined as oscillations of the dependent variables.) Some of the waves supported by the equations are:

- External (surface) gravity waves
- Internal gravity waves
- Inertia-gravity waves
- Acoustic waves (including Lamb waves)
- Rossby waves
- Kelvin waves
- Kelvin-Helmholtz waves

Some of these waves are important for the dynamics of synoptic scale systems, while others are merely "noise." In order to understand dynamic meteorology, we must understand the waves that can occur in the atmosphere.

BASIC DEFINITIONS

- *amplitude* – half of the difference in height between a crest and a trough.
- *wavelength* (λ) – the distance between crests (or troughs)
- *wave number* (K) – $2\pi/\lambda$; the number of radians in a unit distance in the direction of wave propagation (sometimes the wave number is just defined as $1/\lambda$, in which case it is the number of wavelengths per unit distance.)
    - A higher wave number means a shorter wavelength.
    - Units are radians m^{-1}, or sometimes written as just m^{-1}.
    - We can also define wave numbers along each of the axes.
        - k is the wave number in the x-direction ($k = 2\pi/\lambda_x$).
        - l is the wave number in the y-direction ($l = 2\pi/\lambda_y$).
        - m is the wave number in the z-direction. ($m = 2\pi/\lambda_z$).
    - The wave number vector is given by

      $$\vec{K} \equiv k\hat{i} + l\hat{j} + m\hat{k}$$

      (don't confuse k and $\hat{k}$) and points in the direction of propagation of the wave.
- *angular frequency* (ω) – 2π times the number of crests passing a point in a unit of time.
    - Units are radians s^{-1}, sometimes just written as s^{-1}.
- *phase speed* (c) –the speed of an individual crest or trough.
    - For a wave traveling solely in the x-direction, $c = \omega/k$.

- For a wave traveling solely in the y-direction , $c = \omega/l$.
- For a wave traveling solely in the z-direction , $c = \omega/m$.
- For a wave traveling in an arbitrary direction, $c = \omega/\mathrm{K}$, where K is the *total wave number* given by $\mathrm{K}^2 = k^2 + l^2 + m^2$.
- For a wave traveling in an arbitrary direction, there is a phase speed along each axis, given by $c_x = \omega/k$, $c_y = \omega/l$, and $c_z = \omega/m$. ***Note that these are not the components of a vector!***

$$\vec{c} \neq c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

The phase velocity vector is actually given by

$$\vec{c} = \frac{\omega}{\mathrm{K}^2}\vec{\mathrm{K}} = \frac{\omega}{\mathrm{K}^2}(k\,\hat{i} + l\,\hat{j} + m\,\hat{k}).$$

- The magnitude of the phase velocity (the phase speed) is given by

$$c = \frac{\omega}{K}\ .$$

- *group velocity* ($\boldsymbol{c_g}$) – the velocity at which the wave energy moves. Its components are given by

$$\vec{c}_g = \frac{\partial\omega}{\partial k}\hat{i} + \frac{\partial\omega}{\partial l}\hat{j} + \frac{\partial\omega}{\partial m}\hat{k}$$

  - The magnitude of the group velocity (the group speed) is given by

$$c_g = \frac{\partial\omega}{\partial K}\ .$$

- *dispersion relation* – an equation that gives the angular frequency of the wave as a function of wave number and physical parameters,

$$\omega = F\left(k, l, m,\ physical\ parameters\right).$$

Each wave type has a unique dispersion relation. One of our main goals when studying waves is to determine the dispersion relation.

WAVE DISPERSION

- If the group velocity is the same as the phase speed of the individual waves making up the packet, then the waves are *non-dispersive*.
  - If waves are non-dispersive, then the shape of the wave packet never changes in time.
- If the group velocity is different than the phase speed on the waves making up the packet, then the waves are *dispersive*.
  - If the waves are dispersive, then the shape of the wave packet will change with time.
- Waves are dispersive if the phase velocity is not equal to the group velocity.
- Waves are non-dispersive if the phase velocity is equal to the group velocity.

THE EQUATION FOR A WAVE

The equation for a wave traveling in the positive x direction is

$$u(x,t) = A\sin(k\,x - \omega\,t) + B\cos(k\,x - \omega\,t)$$

An alternate way of writing this is

$$u(x,t) = A\sin k(x - ct) + B\cos k(x - ct)$$

For a wave traveling in the negative x direction, the equation is

$$u(x,t) = A\sin(k\,x + \omega\,t) + B\cos(k\,x + \omega\,t)$$

EULER'S FORMULA

Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

From Euler's formula we have the following two identities:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = -i\,\frac{e^{i\theta} - e^{-i\theta}}{2}$$

Using Euler's formula a wave traveling in the positive *x*-direction can be written as

$$u(x,t) = Ae^{i(kx-\omega t)}$$

a wave traveling in the negative *x*-direction can be written as

$$u(x,t) = Ae^{i(kx+\omega t)},$$

where the amplitude *A* may itself be a complex number,

$$A = a_r + ia_i,$$

and gives information about the phase of the wave.

We will frequently use this complex notation for waves because it makes differentiation more straightforward because you don't have to remember whether or not to change the sign (as you do when differentiating sine and cosine functions).

The complex amplitude, *A*, gives information about the phase of the wave. In this form we have the following phase relations between two waves (*u* and *v*), given by

$$u = Ae^{i(kx-\omega t)}$$
$$v = Be^{i(kx-\omega t)}$$

$$u \propto v \quad \text{in phase}$$
$$u \propto -iv \quad 90^\circ \text{ out of phase}$$
$$u \propto -v \quad 180^\circ \text{ out of phase}$$
$$u \propto iv \quad 270^\circ \text{ out of phase}$$

SPECTRAL ANALYSIS

It is rare to find a wave of a single wavelength in the atmosphere. Instead, there are many waves of different wavelengths superimposed on one another. However, we can use the concept of spectral analysis to isolate and study individual waves, recognizing that we can later sum them up if need be. So, keep in mind that real atmospheric disturbances are a collection of many individual waves of differing wavelengths.

Fourier Series – Applies to Continuous, Periodic Functions

Most continuous periodic functions (period = L) can be represented by an infinite sum of sine and cosine functions as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right)$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)dx$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx\,.$$

The Fourier coefficients give the amplitudes of the various sine and cosine waves needed to replicate the original function.

- The coefficient a_0 is just the average of the function.
- The coefficients a_n are the coefficients of the cosine waves (the even part of the function).
- The coefficients b_n are the coefficients of the sine waves (the odd part of the function).

For a completely even function, the b_n's would all be zero, while for a completely odd function, the a_n's would be zero.

Fourier series can also be represented using complex notation, and in this notation

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n \exp\left[\frac{i2\pi nx}{L}\right]$$

where the coefficients α_n are complex numbers, with the real part representing the amplitudes of the cosine waves, and the imaginary part representing the amplitudes of the sine waves,

$$\alpha_n = \frac{1}{2}\left(a_n - ib_n\right) = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\exp\left[\frac{-i2\pi nx}{L}\right]dx\,.$$

Each of the Fourier coefficients, α_n, are associated with a sinusoidal wave of a certain wavelength. If the original function contained one pure wave, then there would

only be two Fourier coefficients (a_1 and b_1). The more sinusoids (more wave numbers) needed to represent the function, the more Fourier coefficients are necessary.
In general:

- Smoother functions require fewer waves to recreate, and have fewer higher frequency components.
- Sharper functions require more waves to recreate, and have more higher frequency components.
- Broad functions require fewer waves to recreate, and have fewer higher frequency components.
- Narrow functions require more waves to recreate, and have more higher frequency components.

Fourier Transforms – Applies to Continuous, Aperiodic Functions

Fourier analysis can be extended to functions that are continuous, but not periodic (aperiodic functions). This is done by representing the function as an infinite integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp[ikx]dk \qquad \textbf{(1)}$$

where the Fourier coefficients are represented by F(k), which is a complex number given by

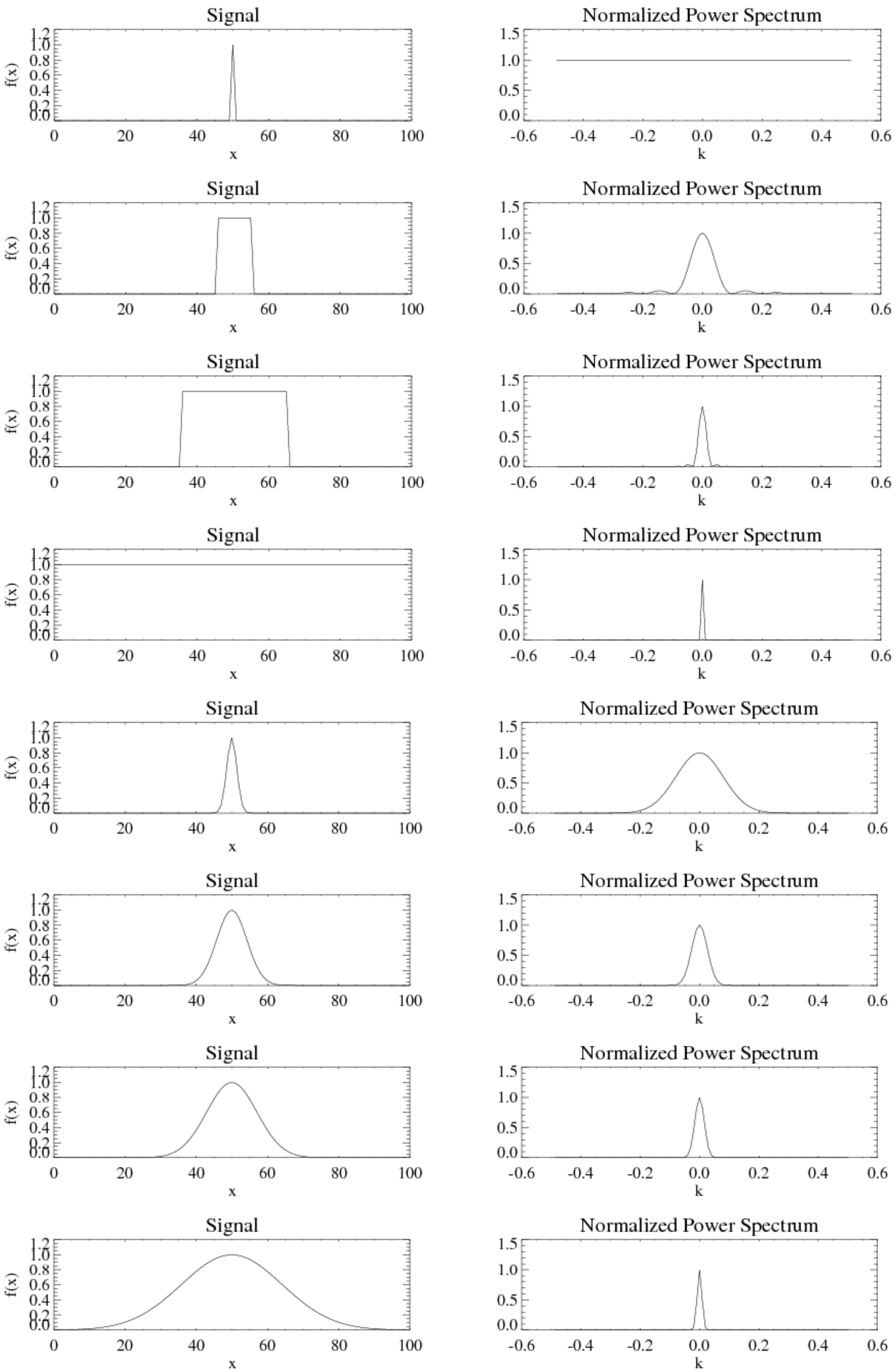
$$F(k) = \int_{-\infty}^{\infty} f(x)\exp[-ikx]dx\,. \qquad \textbf{(2)}$$

Equations (1) and (1) are called the Fourier transform pairs. Equation (1) is the representation of the function in "physical" space. Equation (2) is the representation of the function in "frequency" or "wave number" space. As with Fourier series, the real part of the Fourier coefficient, Re[F(k)], represents the cosine, or even part of the function, while the imaginary part, Im[F(k)], represents the sine, or odd part of the function.

FOURIER SPECTRA OF SOME EXAMPLE FUNCTIONS

As mentioned previously, sharp, narrow functions have more and higher frequency waves in their Fourier spectra then do smooth, broad functions. The figures below shows some example functions and their associated Fourier spectra. The first four figures show box functions of various width, while the second four pictures show Gaussian curves of various width. Things to note:

- In general, the narrower the function, the broader the spectrum, and vice versa.
- The power series of a Gaussian curve is also a Gaussian curve.
- An impulse function has an infinitely broad power spectrum, while an infinitely broad function has a single spike for its power spectrum.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EXERCISES

1. Show the following to be true:

$$\cos(kx-\omega t) = \mathrm{Re}\left[e^{i(kx-\omega t)}\right]$$
$$-\cos(kx-\omega t) = \mathrm{Re}\left[-e^{i(kx-\omega t)}\right]$$
$$\sin(kx-\omega t) = \mathrm{Re}\left[-ie^{i(kx-\omega t)}\right]$$
$$-\sin(kx-\omega t) = \mathrm{Re}\left[ie^{i(kx-\omega t)}\right]$$

2. Show the following to be true:

$$\frac{\partial}{\partial x}e^{i(kx-\omega t)} = ike^{i(kx-\omega t)}$$
$$\frac{\partial^2}{\partial x^2}e^{i(kx-\omega t)} = -k^2e^{i(kx-\omega t)}$$
$$\frac{\partial}{\partial t}e^{i(kx-\omega t)} = -i\omega\, e^{i(kx-\omega t)}$$
$$\frac{\partial^2}{\partial t^2}e^{i(kx-\omega t)} = -\omega^2\, e^{i(kx-\omega t)}$$

3. A wave is represented in complex notation as

$$u(x,t) = Ae^{i(kx-\omega t)}$$

where $A = 2-3i$. Show that this is equivalent to representing the wave as

$$u(x,t) = 2\cos(kx-\omega t) + 3\sin(kx-\omega t).$$

4. Find the phase difference between the following two waves,

$$u(x,t) = Ae^{i(kx-\omega t)}$$
$$v(x,t) = Be^{i(kx-\omega t)}$$

for the following values of *A* and *B*.

a. $A = 2+3i; \quad B = -3+2i$

b. $A = 2+3i; \quad B = -2-3i$

c. $A = 2+3i; \quad B = 3-2i$

d. $A = 2+3i; \quad B = 4+6i$

e. $A = 2+3i; \quad B = 9-6i$

5. a. Let a wave be represented by

$$u(x) = e^{ikx}.$$

Show that u and du/dx are 270° out of phase.

b. Let a wave be represented by

$$u(x) = \cos kx.$$

Show that u and du/dx are 270° out of phase, which shows the consistency of representing sinusoids using complex notation.

6. A wave traveling in two dimensions is represented as

$$u(x, y, t) = Ae^{i(kx+ly-\omega t)}.$$

Show that

$$\nabla^2 u = -\left(k^2 + l^2\right)u\,,$$

demonstrating the Laplacian of a sinusoidal function is proportional to the negative of the original function.

7. What is the physical meaning of a complex frequency? In other words, if ω has an imaginary part, what does this imply? Hint: Put $\omega = \omega_r + i\omega_i$ into

$$u = e^{i(kx-\omega t)}$$

and see what you get.

8. Start with the definition of group velocity, $\vec{c}_g = \frac{\partial \omega}{\partial k}\hat{i} + \frac{\partial \omega}{\partial l}\hat{j} + \frac{\partial \omega}{\partial m}\hat{k}$, and show that the magnitude of the group velocity (the group speed) is given by $c_g = \frac{\partial \omega}{\partial K}$.