ESCI 340 - Cloud Physics and Precipitation Processes
Lesson 6 - Growth of Cloud Droplets by Diffusion
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References:

A Short Course in Cloud Physics, 3rd ed., Rogers and Yau, Ch. 7

Flux

- A ***flux*** is the amount of something passing through a unit of area in a unit of time.
- The units of flux are the units of whatever is being transported, divided by area and time. Examples are:

  Mass flux: kg m^{-2} s^{-1}

  Energy flux: J m^{-2} s^{-1}

  Particle flux: m^{-2} s^{-1}
- The flux is actually a vector that points in the direction of the transport.
- In component form in Cartesian coordinates the flux vector is

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}. \quad (1)$$

Fick's Laws of Diffusion

NOTE: *My notation for number of particles and number density differs from Rogers and Yau. I use N for number density, and n for number of particles.*

- ***Fick's First Law of Diffusion*** states that for particles the flux is always opposite to the gradient of the particle concentration (given as number density, N, in units of particles per cubic meter).
- Mathematically, Fick's First Law is written as

$$\vec{F} = -D\nabla N, \quad (2)$$

  where D is the ***diffusivity***.

  – Diffusivity has units of m^2 s^{-1}.
- Figure 1 shows a rigid volume with flux vectors entering through the left face and leaving through the right face.

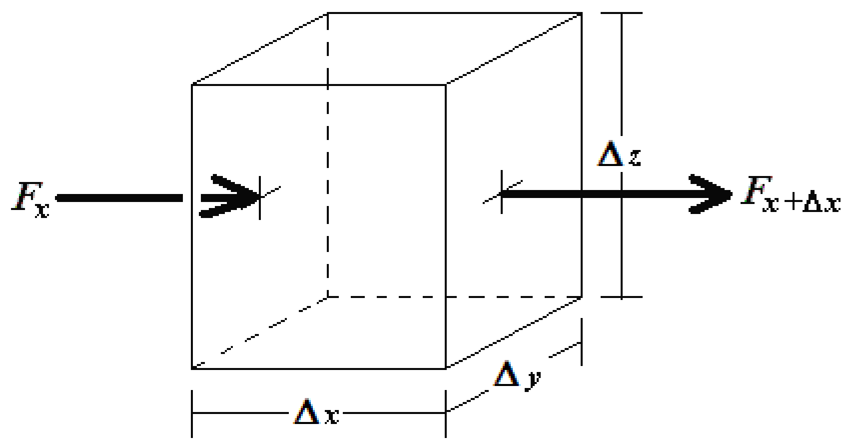

Figure 1: Flux vectors entering and leaving a volume from the x-oriented faces.

- The rate at which particles are entering through the left face is given by $F_x \Delta y \Delta z$.
- The rate at which particles are leaving through the left face is given by $-F_{x+\Delta x} \Delta y \Delta z$.
- The net change in number of particles n in the volume is therefore

$$\frac{\partial n}{\partial t} = -F_{x+\Delta x} \Delta y \Delta z + F_x \Delta y \Delta z = -(F_{x+\Delta x} - F_x) \Delta y \Delta z. \tag{3}$$

- In terms of number density, N, we have

$$\frac{\partial N}{\partial t} = \frac{1}{\Delta x \Delta y \Delta z} \frac{\partial n}{\partial t} = -\frac{1}{\Delta x \Delta y \Delta z}(F_{x+\Delta x} - F_x) \Delta y \Delta z = -\frac{F_{x+\Delta x} - F_x}{\Delta x}. \tag{4}$$

- In the limit as $\Delta x \to 0$, (4) becomes

$$\frac{\partial N}{\partial t} = -\frac{\partial F_x}{\partial x}. \tag{5}$$

- We could write similar expressions to (5) for the y and z directions, and combining the results would yield

$$\frac{\partial N}{\partial t} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z} = -\nabla \cdot \vec{F}. \tag{6}$$

- Equation (6) states that the change in number density in the volume depends on the divergence or convergence of the fluxes.
- If we combine (2) with (6) we get ***Fick's Second Law of Diffusion***,

$$\frac{\partial N}{\partial t} = D \nabla^2 N, \tag{7}$$

which states that the rate of change of number density depends on the Laplacian of the number density.[1]

[1] Equation (7) is a differential equation known as the diffusion equation.

- To get a physical sense of what (7) means, let's look at it in the x-direction only,

$$\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial x^2}. \tag{8}$$

- Figure (2) shows examples of the fluxes in the x-direction and how they relate to the second derivative of N.
  - When $\partial^2 N/\partial x^2 > 0$ there is more flux entering than leaving the box, and so N will increase with time.
  - When $\partial^2 N/\partial x^2 < 0$ there is more flux leaving than entering the box, and so N will decrease with time.
  - When $\partial^2 N/\partial x^2 = 0$ there are equal amounts of flux entering and leaving the box, and so N will not change.

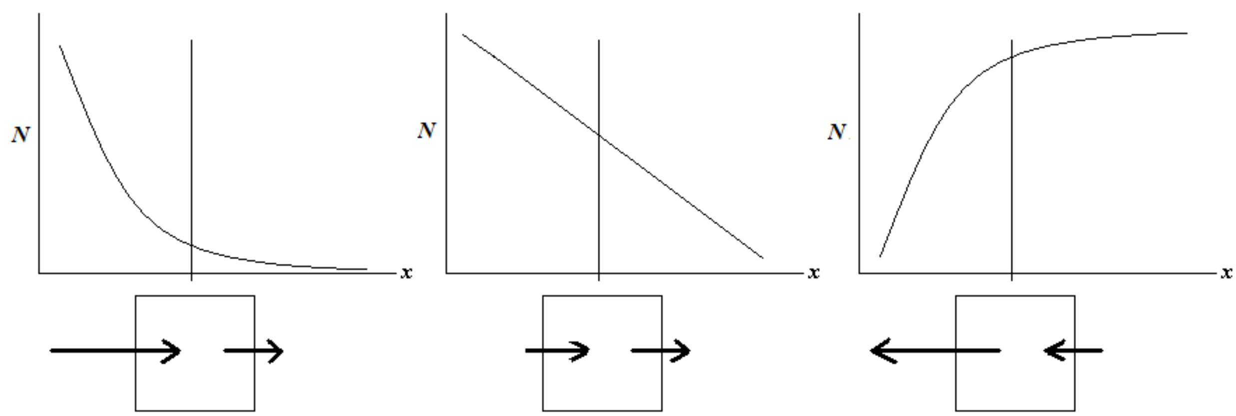

Figure 2: Graph showing the relationship between $\partial^2 N/\partial x^2$ and the flux vectors entering and leaving a volume from the x-oriented faces.

Growth Rate of Droplet by Diffusion

- Once a cloud droplet forms it continues to grow by diffusion of water vapor onto its surface (condensation).
- Figure 3 illustrates a droplet of radius R with radial vapor fluxes at the surface of the droplet denoted by $\vec{F}_R$.

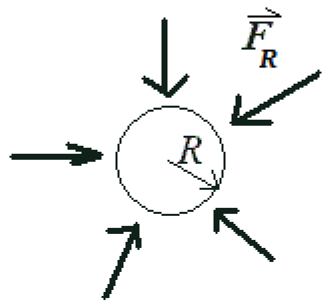

Figure 3: Convergence of radial vapor fluxes, $\vec{F}_R$, at the surface of the droplet results in droplet growth.

- For simplicity we will assume that the fluxes are ***axisymmetric***, meaning that the fluxes only change with distance from the droplet, not with the angle. Another way of saying this is that the fluxes are ***isotropic***.

- If we multiply the flux at the surface of the droplet by the surface area of the droplet we obtain the rate of change of molecules of the droplet,

$$\frac{dn}{dt} = -4\pi R^2 F_R. \tag{9}$$

  - Note that F_R itself is negative, since it is pointing inward toward the droplet. That is why there is a negative in front of (9), so that dn/dt will be positive.

- The flux, F_R, at the surface of the droplet is given by Fick's first law of diffusion, (2), and is $F_R = -\hat{k} \cdot D(\nabla N)_R = -D(\partial N/\partial r)_R$.[2] Therefore (9) becomes

$$\frac{dn}{dt} = 4\pi D R^2 \left(\frac{\partial N}{\partial r}\right)_R. \tag{10}$$

  - Keep in mind that n is the number of water molecules in the droplet itself, whereas N is the number density of water vapor molecules in the air.

- We find $(\partial N/\partial r)_R$ as follows:

  - We assume that N does not change with time, so that from Fick's second law of diffusion, (7), we have

$$\nabla^2 N = 0. \tag{11}$$

[2] In spherical coordinates $\nabla N = \frac{\hat{i}}{r\sin\varphi}\frac{\partial N}{\partial \theta} + \frac{\hat{j}}{r}\frac{\partial N}{\partial \varphi} + \hat{k}\frac{\partial N}{\partial r}$, where φ is zenith angle and θ is azimuth angle. Fortunately we only need the $\hat{k}$ component.

- In spherical coordinates with the droplet at the origin, and since the vapor concentration is axisymmetric, (11) becomes[3]

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial N}{\partial r}\right) = 0. \tag{12}$$

- Integrating (12) twice with respect to r results in

$$N(r) = -\frac{c_1}{r} + c_2, \tag{13}$$

where c_1 and c_2 are the constants of integration. We find them by applying the boundary conditions

$$\begin{aligned} N(r >> R) &= N_b \\ N(R) &= N_R. \end{aligned} \tag{14}$$

where N_b is the background vapor concentration well away from the droplet.[4]

- Applying the boundary conditions (14) to (13) results in

$$\begin{aligned} c_1 &= (N_b - N_R)R \\ c_2 &= N_b. \end{aligned}$$

- Putting these constants back into (13) results in

$$N(r) = -\frac{(N_b - N_R)R}{r} + N_b. \tag{15}$$

- And finally, by taking $\partial/\partial r$ of (15) and evaluating the result at $r = R$, we get that

$$\left(\frac{\partial N}{\partial r}\right)_R = \frac{N_b - N_R}{R}. \tag{16}$$

• Putting (16) into (10) gives us our growth-rate equation for the droplet,

$$\frac{dn}{dt} = 4\pi D R (N_b - N_R). \tag{17}$$

- If the background vapor concentration is larger than that at the droplet surface, $N_b > N_R$, the droplet will grow due to condensation.

- If the background vapor concentration is smaller than that at the droplet surface, $N_b < N_R$, the droplet will shrink due to evaporation.

[3]In spherical coordinates $\nabla^2 N = \frac{1}{r^2 \sin^2\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial N}{\partial \varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 N}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N}{\partial r}\right)$, but with axisymmetry the only non-zero term is the last one.

[4]Rogers and Yau use n_∞ instead of N_b.

Growth Rate in Terms of Droplet Mass and Radius

- Equation (17) can be converted to an equation for the mass growth rate, dm/dt, as follows:
  - Multiply both sides of (17) by the molar mass of water, M_w, and divide by Avogadro's number, $\mathrm{N_A}$,

$$\frac{M_w}{\mathrm{N_A}}\frac{dn}{dt} = \frac{M_w}{\mathrm{N_A}} 4\pi DR(N_b - N_R). \tag{18}$$

  - Since mass is

$$\frac{M_w}{\mathrm{N_A}} n = m$$

    and absolute humidity is

$$\frac{M_w}{\mathrm{N_A}} N = \rho_v,$$

    (18) becomes

$$\frac{dm}{dt} = 4\pi DR(\rho_{vb} - \rho_{vR}). \tag{19}$$

- What would be most convenient is to have an equation for the growth-rate in terms of the radius of the droplet. We can construct this using the chain rule for derivatives,

$$\frac{dR}{dt} = \frac{dR}{dm}\frac{dm}{dt}. \tag{20}$$

  - The mass of a droplet is

$$m = \frac{4}{3}\pi\rho_l R^3,$$

    so

$$\frac{dR}{dm} = \frac{1}{4\pi\rho_l R^2}. \tag{21}$$

- From (19), (20) and (21) we get

$$R\frac{dR}{dt} = \frac{D}{\rho_l}(\rho_{vb} - \rho_{vR}). \tag{22}$$

Other Equations Needed to Solve for Growth Rate

- Equation (22) gives us the ability to integrate forward in time to find an expression for $R(t)$, the radius of the droplet at any future time t.
  - We do not know what value of ρ_{vR} to use, since this depends on the temperature of the surface of the droplet.
  - However, we can assume that at the surface of the droplet the air is saturated, so that $\rho_{vR} = \rho_{vs}$, where ρ_{vs} is the ***saturation absolute humidity***.

- From the ideal gas law for water vapor

$$\rho_{vR} = \rho_{vs} = \frac{e_s}{R_v T_R} \tag{23}$$

where T_R is the temperature at the surface of the droplet.

  * Note that T_R is not necessarily the same as the air temperature. The droplet warms or cools depending on whether there is condensation or evaporation at the droplet's surface.

- e_s is the saturation vapor pressure over a curved, impure droplet which we know to be

$$e_s = e_0 \left(1 + \frac{a}{R} - \frac{b}{R^3}\right) \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T_R}\right)\right], \tag{24}$$

so that

$$\rho_{vR} = \frac{e_0}{R_v T_R}\left(1 + \frac{a}{R} - \frac{b}{R^3}\right) \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T_R}\right)\right]. \tag{25}$$

- Equations (22) and (25) are two equations, but we have three unknown quantities: R, ρ_{vR}, and T_R. Therefore we still need one more equation in order to have a closed set that we can solve.

- The third equation comes from balancing the gain of latent heat due to condensation with the loss of sensible heat due to thermal diffusivity.

  - The gain of latent heat due to condensation is given by

$$J_{latent} = L_v \frac{dm}{dt} = 4\pi R L_v D(\rho_{vb} - \rho_{vR}). \tag{26}$$

  - The sensible lost to the air by diffusion is

$$J_{sensible} = -4\pi R K(T_R - T_b), \tag{27}$$

  where K is the thermal diffusivity of air and T_b is the temperature of the air.

  - The sensible heat loss and latent heat gain must sum to zero in equilibrium, and so summing (26) and (27) and setting them equal to zero results in

$$\rho_{vb} - \rho_{vR} = \frac{K}{L_v D}(T_R - T_b). \tag{28}$$

Calculations of Growth Rates

- Equations (22), (25) and (28) are three equations for three unknown quantities, R, ρ_{vR}, and T_R. The equations are rewritten here,

$$\begin{aligned} R\frac{dR}{dt} &= \frac{D}{\rho_l}(\rho_{vb} - \rho_{vR}) \\ \rho_{vR} &= \frac{e_0}{R_v T_R}\left(1 + \frac{a}{R} - \frac{b}{R^3}\right) \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T_R}\right)\right] \\ \rho_{vb} - \rho_{vR} &= \frac{K}{L_v D}(T_R - T_b). \end{aligned}$$

- We can solve these three equations to find the growth rate and radius of a droplet at any future time, t.
- However, the equations are quite complex and cannot be solved analytically. They need to be solved numerically.
- A somewhat simplified, though not as accurate, set of growth equations is[5]

$$R\frac{dR}{dt} = \frac{S - 1 - \frac{a}{R} + \frac{b}{R^3}}{F_k + F_d} \tag{29}$$

$$F_k = \left(\frac{L_v}{R_v T_b} - 1\right)\frac{L_v \rho_l}{K T_b} \tag{30}$$

$$F_d = \frac{\rho_l R_v T_b}{D e^*_{s\infty}}, \tag{31}$$

  where the saturation vapor pressure used in calculating F_d is that for a flat surface of pure water.

- These equations still need to be integrated numerically. The result for a droplet starting at radius $r_0 = 0.75\mu$m is shown in Fig. 4.
- Note that after 20 hours the droplet is still only has a radius slightly larger than 60 μm.

[5]These equations are developed in *The Physics of Clouds* by Mason (1971).

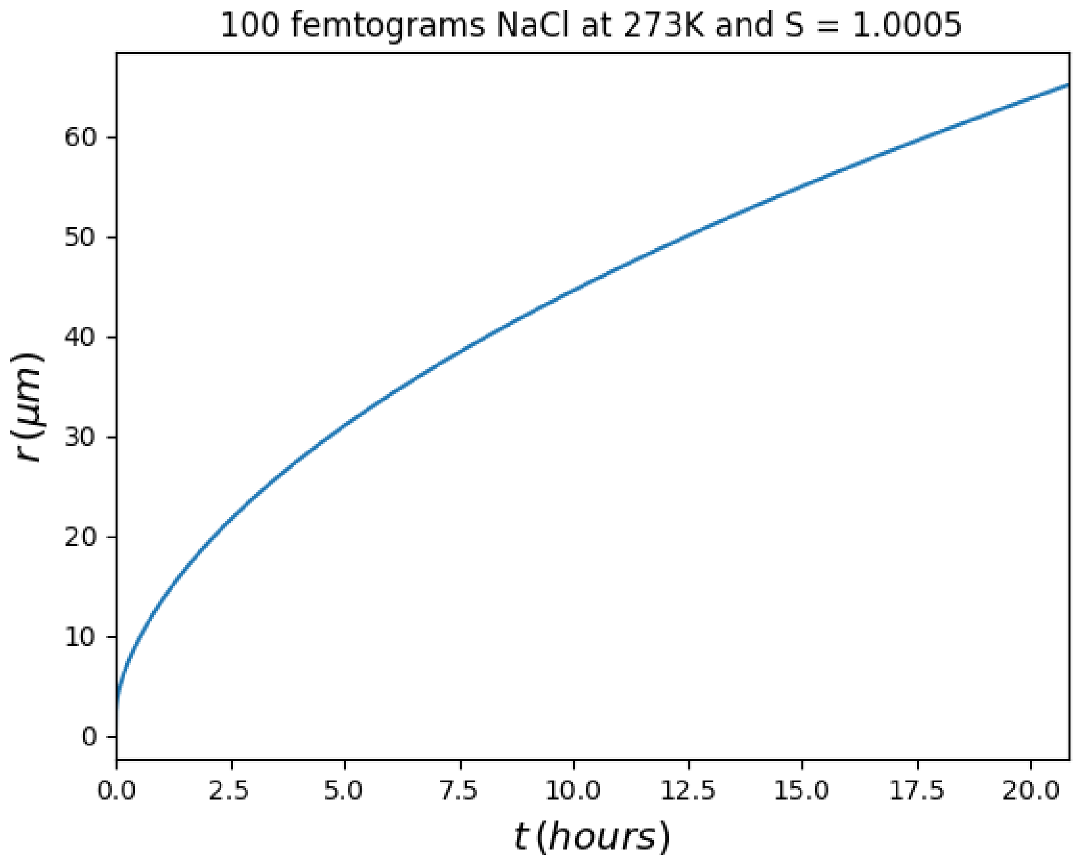

Figure 4: Growth of droplet initially of radius 0.75 μm for a solute of 100 femtograms of NaCl.

- Fig. 5 shows the effect of doubling the mass of solute. Although the droplet initially grows faster with more solute, the growth rates quickly become the same.

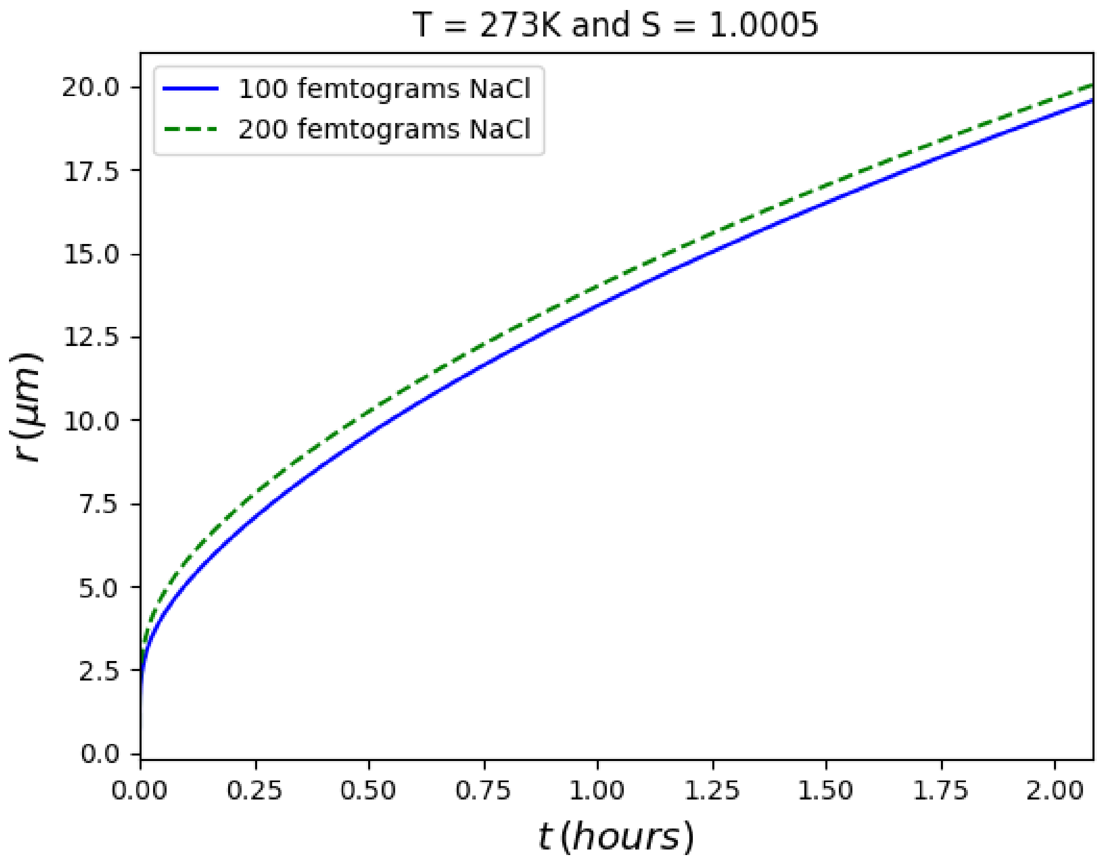

Figure 5: Growth of droplet initially of radius 0.75 μm for two different solute masses.

Final Comments on Diffusionial Growth

- In order to be large enough to fall fast enough to reach the ground without evaporating, a droplet has to reach a size of at least 0.1 mm in diameter (0.05 mm or 50 μm in radius).
- A typical raindrop has a diameter of 2 mm (radius of 1 mm, or 1000 μm).
- Clouds can form and rain start to fall in a matter of 30 minutes or so.
- Diffusional growth explains how very tiny, brand-new cloud droplets grow to typical cloud droplet sizes, but is too slow to explain how precipitation forms.

Exercises

1. For radii larger than a few microns the curvature and solute effects become negligible, and (29) becomes

$$R\frac{dR}{dt} = \frac{S-1}{F_k + F_d}. \tag{32}$$

(a) Show that (32) can be analytically integrated to obtain

$$R(t) = \sqrt{R_0^2 + 2\xi t} \tag{33}$$

where $\xi = (S-1)/(F_k + F_d)$.

(b) Use (33) to find how long it would take for a droplet to grow from an initial radius of 5 microns to a final radius of 50 microns for a saturation ratio of 1.0005 and a temperature of 273.15K. Use Table 7.1 on page 103 of Rogers and Yau to determine values for K and D. For L_v use 2.5×10^6J kg^{-1}.

2. Show that (22) can be written in terms of vapor pressure rather than absolute humidity

$$R\frac{dR}{dt} = \frac{D}{R_v \rho_l}\left(\frac{e_{sb}}{T_b} - \frac{e_{sR}}{T_R}\right). \tag{34}$$

3. Integrate

$$\frac{d}{dr}\left(r^2 \frac{dN}{dr}\right) = 0$$

and apply the boundary conditions

$$N(r >> R) = N_b$$
$$N(R) = N_R$$

to show that

$$N(r) = -\frac{(N_b - N_R)R}{r} + N_b.$$