ESCI 340 - Cloud Physics and Precipitation Processes
Lesson 7 - Growth by Collision-Coalescence
Dr. DeCaria

References:

A Short Course in Cloud Physics, 3rd ed., Rogers and Yau, Ch. 8

Terminal Velocity

- The equilibrium fall speed of a droplet (***terminal velocity*** is a balance between the frictional drag and the weight of the droplet.

  Friction: Frictional force is given by

$$F_r = \frac{\pi}{2} r^2 u^2 \rho C_D, \tag{1}$$

  where u is the fall speed and C_D is the ***drag coefficient***.

  - The drag coefficient itself may be a function of the fall speed.

  Weight: The weight of the droplet is given by

$$F_g = \frac{4}{3}\pi r^3 g \rho_l. \tag{2}$$

- Setting (1) equal to (2) and solving for the fall speed leads to the formula for the terminal velocity of the droplet

$$u = \sqrt{\frac{8}{3}\frac{g\rho_l r}{\rho C_D}}. \tag{3}$$

- Table 1 shows empirical formulas for terminal velocity for droplets in varying size ranges.

Radius	Terminal Velocity	Parameter Values
$r < 40\ \mu$m	$u = ar^2$	$a = 1.19 \times 10^6$ cm^{-1} s^{-1}
$40\ \mu\text{m} \le r < 0.6$ mm	$u = br$	$b = 8 \times 10^3$ s^{-1}
$r \ge 0.6$ mm	$u = c(\rho_0/\rho)r^{1/2}$	$c = 2.2 \times 10^3$ cm$^{1/2}$ s^{-1}; $\rho_0 = 1.20$ kg m^{-3}

Table 1: Empirical formulas for terminal velocity for different droplet sizes.

Growth Due to Droplet Collisions

- Figure 1 illustrates the volume swept out by a large droplet of radius R falling through a population of smaller droplets of radius r.

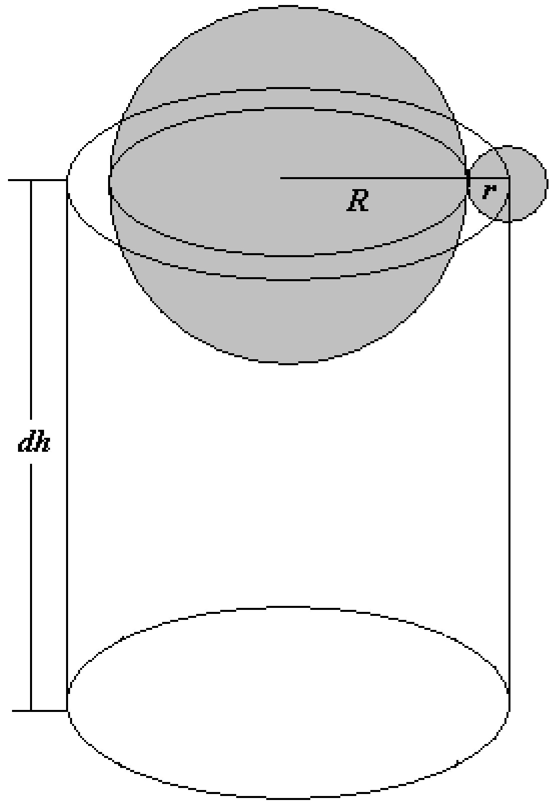

Figure 1: Collection volume swept out by a large droplet of radius R in a population of smaler droplets of radius r.

- The collection volume is given by

$$dV = \pi(R + r)^2 dh. \tag{4}$$

- If the larger droplet is falling at terminal velocity $u(R)$, then $dh = u(R)dt$, and so (4) becomes

$$dV = \pi(R + r)^2 u(R)dt. \tag{5}$$

- The large droplet will collide with any smaller droplets that lie within the collection volume. If these smaller droplets all stick to the larger droplet, the larger droplets mass will increase by the liquid water content, M, times the collection volume. So, the mass-growth rate of the larger droplet will be

$$dm = MdV = \pi(R + r)^2 Mu(R)dt. \tag{6}$$

- Equation (6) assumes that the smaller droplets are stationary. But in reality they are also falling at their terminal velocity, $u(r)$. This decreases the 'effective' volume through which the larger drop sweeps. We account for this by using the relative velocity between the large and small droplets, $u(R) - u(r)$ in (6) to get

$$dm = \pi(R + r)^2 M[u(R) - u(r)]dt, \tag{7}$$

which on dividing by dt gives our mass-growth-rate equation

$$\frac{dm}{dt} = \pi(R+r)^2 M[u(R) - u(r)]. \tag{8}$$

Correction for Collection Efficiency

- In the derivation of (8) we assumed that any small droplet that are within the effective volume swept by the larger droplet would adhere, or ***coalesce*** to the larger droplet.
- In reality, not all collisions result in coalescence. This is due to several factors:
  - A smaller droplet that is in the path of the larger droplet may get pushed out of the way by the air currents around the larger droplet. Think of a bug heading for your windshield. Sometimes the air currents force the bug up and over your car, sparing the bug a gruesome fate.
  - Droplets that weren't even in the path of the larger droplet may get pulled in from behind the droplet. This is called ***wake capture***. Think of leaves following behind a moving car in the autumn.
  - Even if two droplets do collide, it is not guaranteed that they will coalesce. There may be a microfilm of air preventing the two droplets from sticking, or the surface tension of the two droplets may just allow them to bounce off each other.
- To account for all these factors we incorporate a ***collection efficiency***, E, into (8), which then becomes

$$\frac{dm}{dt} = \pi EM(R+r)^2[u(R) - u(r)]. \tag{9}$$

  - The collection efficiency is a dimensionless number, usually less than 1, although technically it could exceed 1 if wake capture is the only or the dominant factor.
  - The collection efficiency is empirically determined.
- Equation (9) if often written as

$$\frac{dm}{dt} = K(R,r)M, \tag{10}$$

where

$$K(R,r) = \pi E(R+r)^2[u(R) - u(r)], \tag{11}$$

and is called the ***gravitational collection kernel***.

Growth Rate in Terms of Radius

- We can create a growth-rate equation in terms of radius by using the chain rule,

$$\frac{dR}{dt} = \frac{dR}{dm}\frac{dm}{dt}.$$

- Since $m = \frac{4}{3}\pi\rho_l R^3$, then $\frac{dR}{dm} = (4\pi\rho_l R^2)^{-1}$. So, in terms of radius our growth-rate equation is

$$\frac{dR}{dt} = \frac{EM}{4\rho_l}\frac{(R+r)^2}{R^2}[u(R) - u(r)]. \tag{12}$$

- We can further simplify this result if we assume that

$$u(R) >> u(r)$$
$$R >> r$$

allowing us to ignore r and $u(r)$ in (12). The simplified expression for growth rate is then

$$\frac{dR}{dt} = \frac{EM}{4\rho_l}u(R). \tag{13}$$

- If we assume a terminal velocity that is linear with radius,[1] $u(R) = bR$, where b is the value from Table 1, then (13) can be integrated analytically to determine the droplet radius as a function of time. The results are shown in Fig. 2 for a droplet of initial radius 40 μm and a collection efficiency of 90%.

[1]This is really only valid for 40 μm $\leq r <$ 0.6 mm, but we will use it here for even larger drops for illustration purposes.

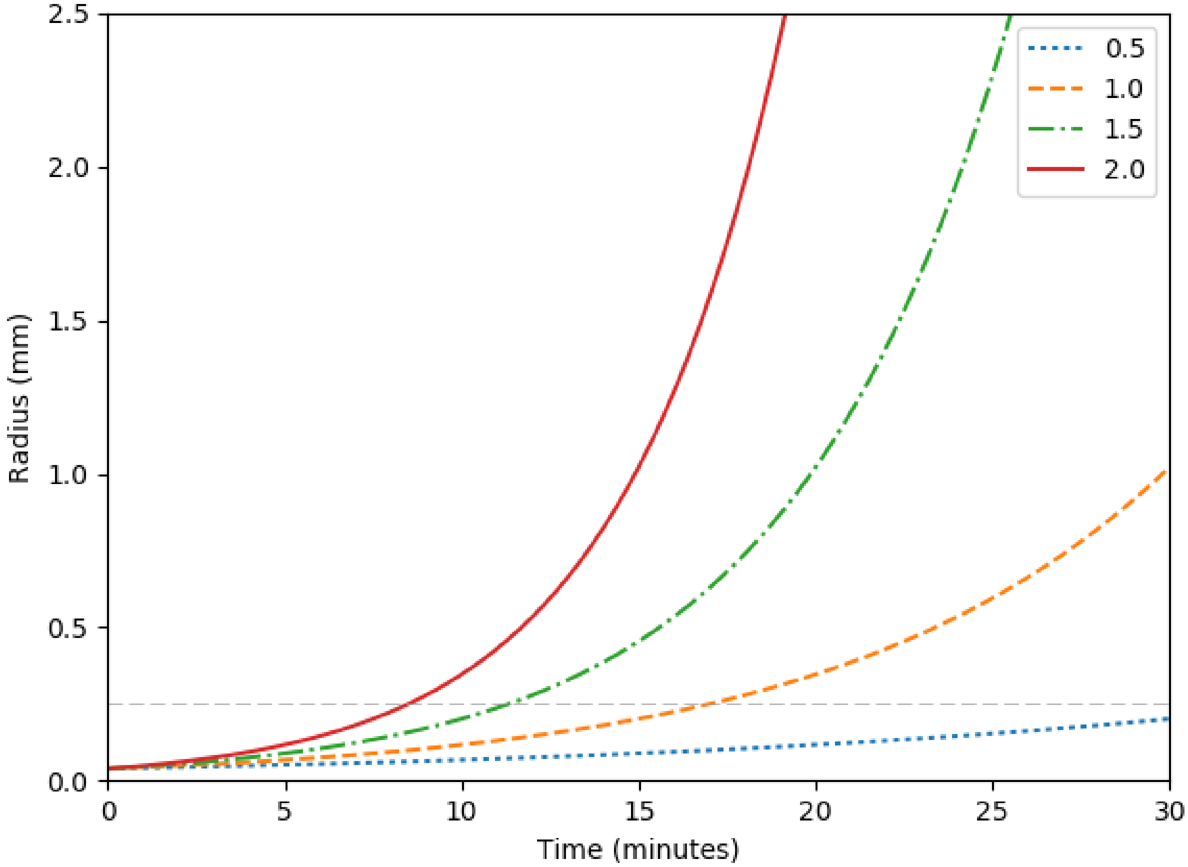

Figure 2: Growth rates from (13) for different liquid water contents (g m^{-3}) for initial droplet radius of 40 μm and a collection efficiency of 90%. The faint, dashed horizontal line represents the cut-off radius between drizzle and rain.

Drop Radius vs. Altitude

- Using the chain rule (13) can be converted to an equation for droplet radius as a function of height,

$$\frac{dR}{dt} = \frac{dR}{dz}\frac{dz}{dt}. \tag{14}$$

- dz/dt is the vertical velocity of the drop, which is its terminal velocity subtracted from the updraft speed, $U - u(R)$. Therefore

$$\frac{dR}{dz} = \frac{1}{U - u(R)}\frac{dR}{dt} = \frac{EM}{4\rho_l}\frac{u(R)}{U - u(R)}. \tag{15}$$

- Allowing $u(R) = bR$ and integrating (15) gives an equation for droplet altitude versus radius,

$$z = z_0 + \frac{4U\rho_l}{bEM}\left[\ln\left(\frac{R}{R_o}\right) - \frac{b}{U}(R - R_0)\right], \tag{16}$$

where R_0 is the initial droplet radius, and z_0 is the altitude at cloud base.

- Figure 3 shows the results for various updraft speeds.

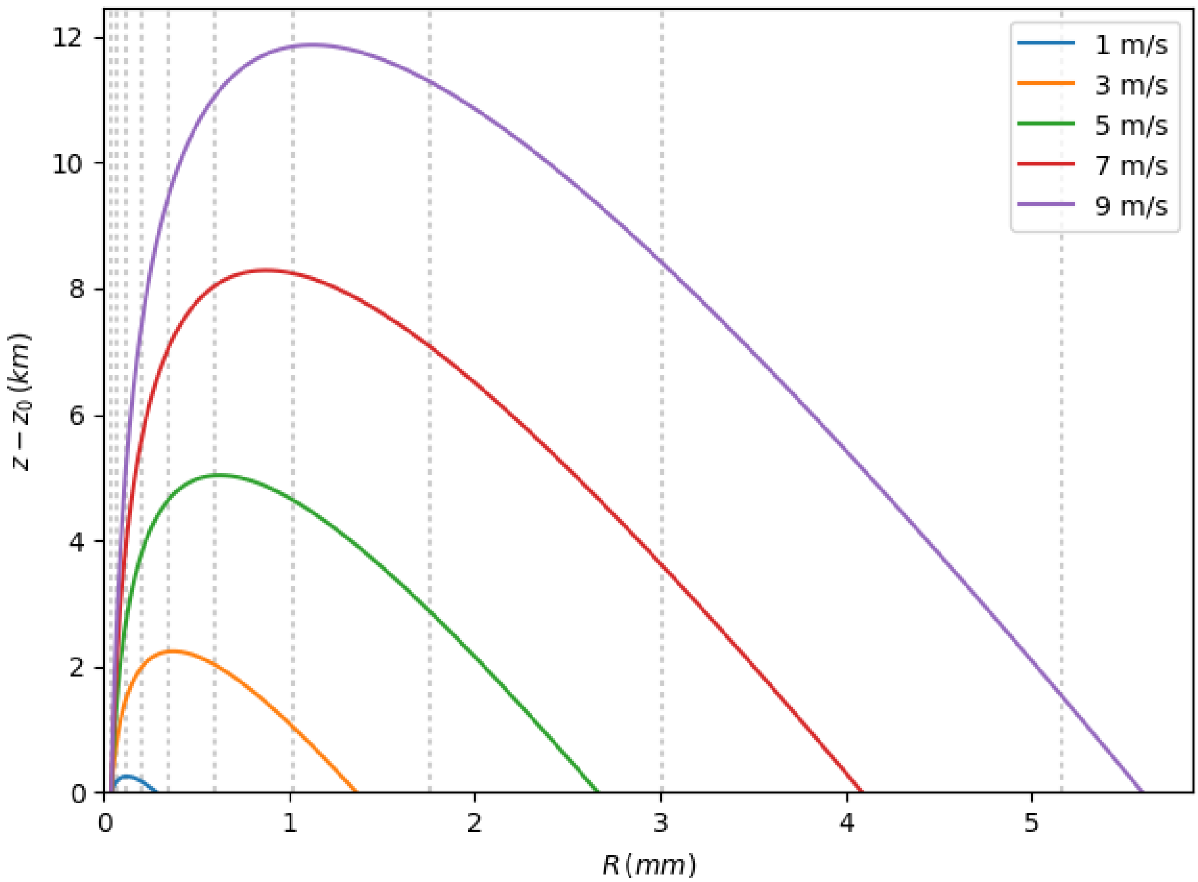

Figure 3: Droplet altitude vs. radius for varying values of updraft velocity, U, from Eqn. (16), with $E = 0.9$, $M = 1$ g m^{-3}, and $R_0 = 40$ μm. Faint dotted vertical lines are at 5 minute intervals.

Droplet Trajectories

- We can derive an equation for the trajectory of a droplet within the cloud by first starting with the expression

$$\frac{dz}{dt} = U - u(R). \tag{17}$$

  - What we need is an expression for the terminal velocity as a function of time, $u(t)$. We can derive this by the chain rule,

$$\frac{du}{dt} = \frac{du}{dR}\frac{dR}{dt}. \tag{18}$$

– Assuming the terminal velocity is given by $u = bR$, and using (13) for dR/dt, (18) becomes

$$\frac{du}{dt} = \frac{bEM}{4\rho_l}u. \tag{19}$$

– Integrating (19) results in

$$u(t) = bR_0 \exp\left(\frac{bEM}{4\rho_l}t\right), \tag{20}$$

where $bR_0 = u_0$ is the terminal velocity at $t = 0$.

- Putting (20) into (17) and integrating again with respect to t results in

$$z(t) = z_0 + Ut + \frac{4\rho_l R_0}{EM}\left[1 - \exp\left(\frac{bEM}{4\rho_l}t\right)\right], \tag{21}$$

where z_0 is the altitude at cloud base.

- Figure 4 shows results for varying values of updraft speed, U.
  – The stronger the updraft, the longer the droplet remains within the cloud.
  – Stronger updrafts also lift the droplet higher into the cloud.

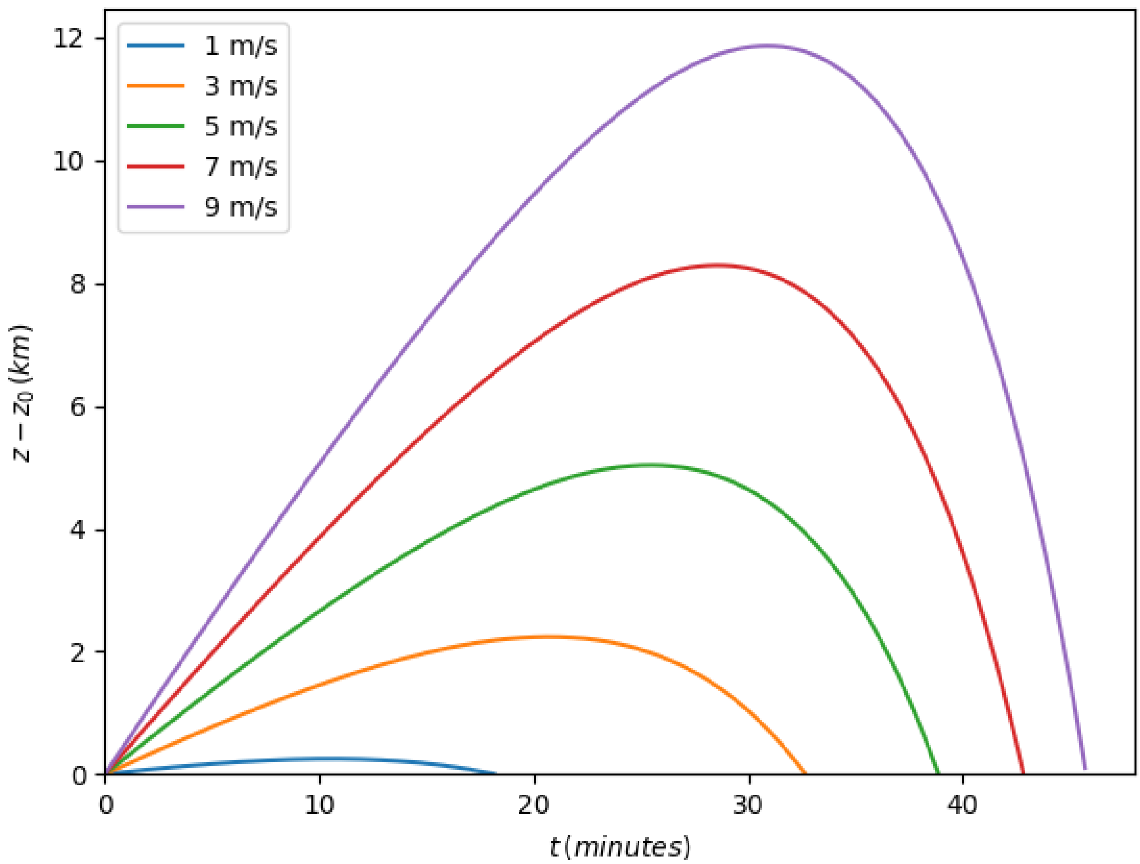

Figure 4: Droplet trajectories for varying values of updraft velocity, U, from Eqn. (21), with $E = 0.9$, $M = 1$ g m^{-3}, and $R_0 = 40$ μm.

Nonuniform Droplet Populations

- The equations we've developed so far only apply to a droplet that is falling through a ***uniform population*** of smaller droplets (i.e., the smaller droplets are all the same size).
- Equation (8) is easily extended to a population of smaller droplets of N different sizes by using a summation

$$\frac{dm}{dt} = \pi \sum_{i=1}^{N} E_i M_i (R + r_i)^2 \left[u(R) - u(r_i)\right]. \tag{22}$$

  - Each of the N different droplet sizes has its own radius, collection efficiency, and liquid water content, represented as r_i, E_i, and M_i.

- If we define the collection kernal for each droplet size as

$$K(R, r_i) = \pi E_i (R + r_i)^2 \left[u(R) - u(r_i)\right] \tag{23}$$

then (22) becomes

$$\frac{dm}{dt} = \sum_{i=1}^{N} K(R, r_i) M_i. \tag{24}$$

- If the droplet population is a continuous spectrum, then the summation in (24) becomes an integral,

$$\frac{dm}{dt} = \int K(R, r) dM, \tag{25}$$

and we know that

$$dM = \frac{4}{3}\pi\rho_l r^3 n_d(r) dr.$$

Therefore, (25) becomes

$$\frac{dm}{dt} = \frac{4}{3}\pi\rho_l \int_0^\infty K(R, r) r^3 n_d(r) dr. \tag{26}$$

- Equation (26) must be solved numerically, and is complicated by the fact that the collection kernel has to remain within the integral.

- The collection kernel is usually determined empirically.

- If we assume thath $R >> r$ and $u(R) >> u(r)$ then we can show (see Exercises) that

$$\frac{dR}{dt} = \frac{\pi u(R)}{3} \int_0^\infty E r^3 n_d(r) dr. \tag{27}$$

Using the mean-value theorem this becomes

$$\frac{dR}{dt} = \frac{\pi u(R)}{3} \overline{E} \int_0^\infty r^3 n_d(r) dr, \tag{28}$$

where

$$\overline{E} = \frac{4\pi\rho_l}{3M} \int_0^\infty E r^3 n_d(r) dr \tag{29}$$

is the ***mass-weighted mean*** collection efficiency.[2]

- Equation (28) can also be written as

$$\frac{dR}{dt} = \frac{\overline{E}M}{4\rho_l} u(R), \tag{30}$$

which is nearly identical to (13).

[2] A mass-weighted mean is the same a volume-weighted mean, since mass and volume are related through the density of liquid water, which is essentially a constant.

- This is nice, because all the analysis and graphs that we created based on a uniform droplet population are still relevant for a continuous droplet population, as long as we used the mass-weighted mean collection efficiency calculated from (29).

Further Comments on Collision-Coalescence

- The growth rate is very sensitive to the drop size spectrum, with broader spectra leading to faster growth rates.
- A broad spectrum is beneficial because it leads to greater relative velocity between the drops, and more chance for collision. In a narrow spectrum, all the drops are falling at roughly the same velocity, and are less likely to collide.
- Clouds droplets are initially formed via diffusion, which leads to a narrowing, rather than a broadening of the drop size spectra.
- Computations using (26) with typical drop size spectra give growth rates too small to explain how precipitation sized drops form in reality.
- To achieve growth rates comparable to that observed in natural clouds (about 15 minutes), a stochastic process is required, whereby a small population of fortunate drops happen to grow much faster than the average rate.
- As these drops grow very large (4 to 5 mm), they become unstable and break apart. This creates some additional large drops which can themselves start to grow through collision-coalescence.
- The physical cause of these stochastic phenomena is a source of study, but turbulence is likely a factor.
- The collision-coalescence process is often called the ***warm-rain*** process, since it is the only way to explain precipitation formation in clouds that remain above freezing. However, it can also occur in cold clouds.

Exercises

1. Use a terminal velocity that is linear with radius, $u(R) = bR$, in (13) to show that

$$R(t) = R_0 \exp\left(\frac{bEM}{4\rho_l} t\right).$$

2. (a) Use $u(R) = bR$ and perform the integration to derive (16) from (15).

(b) At the top of a droplet's trajectory its terminal velocity is equal to the updraft velocity. Use this to show that from (16) the maximum altitude of a droplet is given by

$$z_{max} = z_0 + \frac{4\rho_l}{EM}\left[\frac{U}{b}\ln\left(\frac{U}{aR_0}\right) - \frac{U}{b} + R_0\right].$$

(c) Integrate (13) to show that the time required for a droplet to reach the top of it's trajectory is

$$t_{up} = \frac{4\rho_l}{bEM}\ln\left(\frac{U}{bR_0}\right).$$

3. A droplet is at the very top of it's trajectory in a cloud of liquid water content M and updraft speed U. The terminal velocity of the drop is given by $u(R) = c\sqrt{R}$. Integrate (13) to show that the droplet radius as a function of time is given by

$$R(t) = \left(\frac{U}{c} + \frac{EMc}{8\rho_l}t\right)^2.$$

4. Equaton (15) is

$$\frac{dR}{dz} = \frac{EM}{4\rho_l}\frac{u(R)}{U - u(R)}.$$

(a) This equation implies that for small droplets dR/dz will be positive, while for large droplets dR/dz will be negative. What is the physical explanation for this?

(b) When the terminal velocity of a droplet equals the updraft velocity then $dR/dz = \infty$. What is the physical significance of this?

5. (a) Show that for a continuous drop-size spectrum that the growth rate equation in terms of radius is

$$\frac{dR}{dt} = \frac{1}{3R^2}\int_0^\infty K(R,r)r^3 n_d(r)dr. \tag{31}$$

(b) Show that if $R >> r$ and $u(R) >> u(r)$ then (31) becomes

$$\frac{dR}{dt} = \frac{\pi u(R)}{3}\int_0^\infty E r^3 n_d(r)dr. \tag{32}$$

(c) Show that (32) is equivalent to

$$\frac{dR}{dt} = \frac{\overline{E}M}{4\rho_l}u(R).$$