

ESCI 485 – Air/sea Interaction  
 Lesson 6 – Wind Driven Circulation  
 Dr. DeCaria

References: *Introductory Dynamical Oceanography*, Pond and Pickard  
*Principles of Ocean Physics*, Apel

**STOMMEL'S SOLUTION FOR WESTWARD INTENSIFICATION**

- The surface circulation in an ocean basin is driven by the wind.
- Characteristic of these circulations (or *gyres*) is that they are anticyclonic.
- A striking feature of these gyres in the strong poleward currents on the western side of the ocean basins (off of the east coast of the continents).
- To try to understand the cause of this westward intensification, Stommel in 1948 proposed the following.
  - Use the shallow water equations of motion for a fluid of constant depth  $H$ . (This presumes that the fluid is barotropic).
  - Let  $f = f_0 + \beta y$
  - Include friction supplied at the bottom of the fluid. The friction is assumed to be linear with speed.

$$\begin{aligned} \frac{\partial u}{\partial t} + \vec{V} \cdot \nabla u &= -g \frac{\partial \eta}{\partial x} + (f_0 + \beta y)v - \gamma u + \frac{\tau_x}{\rho H} \\ \frac{\partial v}{\partial t} + \vec{V} \cdot \nabla v &= -g \frac{\partial \eta}{\partial y} - (f_0 + \beta y)u - \gamma v + \frac{\tau_y}{\rho H} \end{aligned} \quad (1)$$

- Form a vorticity equation by cross-differentiating these equations

$$\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial x} \frac{\partial v}{\partial t} - \frac{\partial}{\partial y} \frac{\partial u}{\partial t}. \quad (2)$$

- The equation formed in this way is

$$\frac{\partial \zeta}{\partial t} = \vec{V} \cdot \nabla \zeta - \beta v - \gamma \zeta + \frac{1}{\rho H} \nabla \times \vec{\tau}$$

- This vorticity equation says that the relative vorticity of a fluid parcel can be changed via three ways:
  - Moving the parcel north or south (the *beta* effect). Since absolute vorticity is conserved, moving the parcel north toward higher planetary vorticity will decrease the relative vorticity.

- Friction – The friction tries to move the vorticity toward zero. If the vorticity is positive, friction will decrease it. If the vorticity is negative, friction will increase it.
- Wind stress – The curl of the wind stress will add vorticity if the curl is positive, or decrease vorticity if the curl is negative.
- If the flow is non-divergent the velocity components can be given in terms of a stream function,  $\psi$ , such that

$$u = -\frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \psi}{\partial x}$$

$$\zeta = \nabla^2 \psi \quad .$$

This lets us write the vorticity equation as

$$\frac{\partial}{\partial t} \nabla^2 \psi = -\vec{V} \cdot \nabla (\nabla^2 \psi) - \beta \frac{\partial \psi}{\partial x} - \gamma \nabla^2 \psi + \frac{1}{\rho H} \nabla \times \vec{\tau} \quad . \quad (3)$$

- Stommel looked at the steady state solution to Eq. (3), so that the local time derivative could be ignored. He also assumed the advection term was negligible in the total derivative. This simplified the equation to

$$\nabla^2 \psi + \frac{\beta}{\gamma} \frac{\partial \psi}{\partial x} = \frac{1}{\gamma \rho H} \nabla \times \vec{\tau} \quad . \quad (4)$$

- Stommel assumed a simplistic form for the wind stress, given as

$$\tau_x = -\tau_0 \cos\left(\frac{\pi y}{y_n}\right)$$

$$\tau_y = 0$$

so that

$$\nabla \times \vec{\tau} = -\frac{\pi \tau_0}{y_n} \sin\left(\frac{\pi y}{y_n}\right) \quad .$$

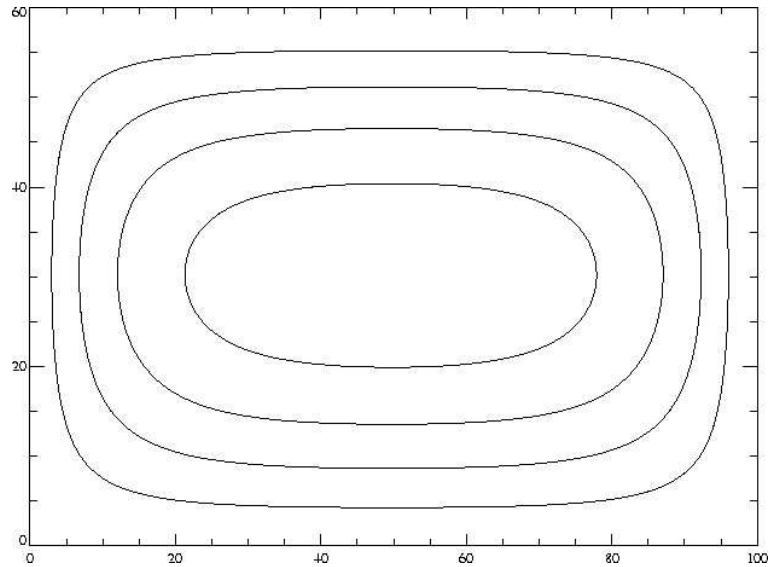
The equation for the stream function is then

$$\nabla^2 \psi + \frac{\beta}{\gamma} \frac{\partial \psi}{\partial x} = -\frac{\pi \tau_0}{\gamma \rho H y_n} \sin\left(\frac{\pi y}{y_n}\right) \quad . \quad (5)$$

- Stommel investigated the solution of Eq. (5) for a rectangular ocean of constant depth for the case of a rotating Earth for two cases: constant Coriolis parameter

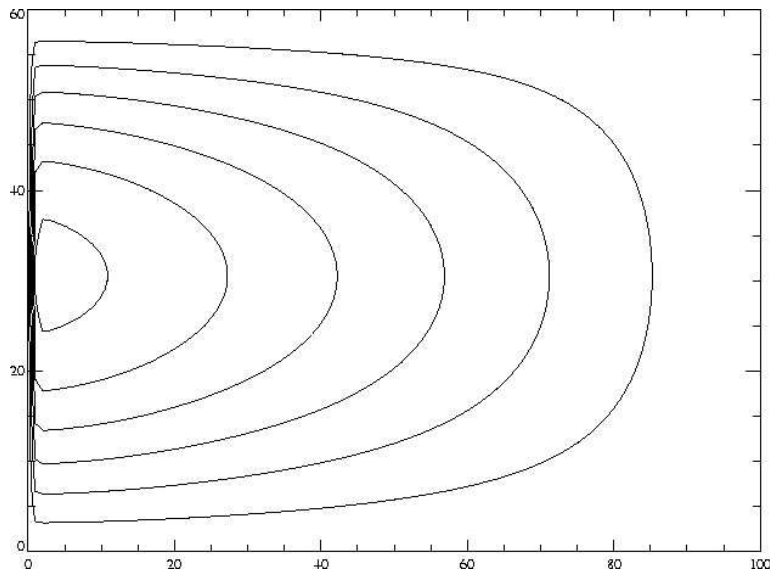
(the  $f$ -plane approximation;  $\beta = 0$ ); Coriolis parameter changing with latitude at a constant rate (the  $\beta$ -plane approximation).

- On the  $f$ -plane, the solution for the streamfunction looks like



- There is no westward intensification in this case.

- On the  $\beta$ -plane the solution for the streamfunction looks like



- The change in Coriolis parameter with latitude ( $\beta$ ) is the key to westward intensification.

## MUNK'S SOLUTION FOR WESTWARD INTENSIFICATION

- Stommel's solution, though correctly allowing a western boundary current, was very simplistic.

- Stommel assumed a barotropic ocean with friction supplied at the bottom.

In the real ocean most of the current is in the upper layers, and there is very little flow at the bottom.

- Munk (1950) used the more complete equations of motion and a more realistic friction term due to turbulent fluxes (Reynolds stresses). These equations are

$$\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + (f_0 + \beta y)v + A_h \nabla_h^2 u - \frac{1}{\rho} \frac{\partial \tau_{rx}}{\partial z} \\ \frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - (f_0 + \beta y)u + A_h \nabla_h^2 v - \frac{1}{\rho} \frac{\partial \tau_{ry}}{\partial z}\end{aligned}\quad (6)$$

where  $\tau_{rx}$  and  $\tau_{ry}$  are the vertical Reynolds stresses.

- A vorticity equation is again formed by cross differentiation, and gives

$$\frac{\partial \zeta}{\partial t} = -\vec{v} \cdot \nabla \zeta - \beta v + A_h \nabla_h^2 \zeta - \frac{1}{\rho} \frac{\partial}{\partial z} (\nabla \times \vec{\tau}_r) \quad (7)$$

- Like Stommel, Munk was interested in the steady state solution. He also assumed that the advective terms were negligible. Therefore his equation becomes.

$$0 = -\beta v + A_h \nabla_h^2 \zeta - \frac{1}{\rho} \frac{\partial}{\partial z} (\nabla \times \vec{\tau}_r).$$

- Integrating through the depth of the ocean yields

$$-\beta \int_{-H}^0 v dz + A_h \nabla_h^2 \int_{-H}^0 \zeta dz - \frac{1}{\rho} (\nabla \times \vec{\tau}) = 0.$$

- Using the streamfunction we get

$$-\beta \int_{-H}^0 \frac{\partial \psi}{\partial x} dz + A_h \nabla_h^2 \int_{-H}^0 \nabla^2 \psi dz - \frac{1}{\rho} (\nabla \times \vec{\tau}) = 0.$$

- Defining the mass transport stream function as

$$\psi_M \equiv \rho \int_{-H}^0 \psi dz \quad (8)$$

we end up with a fourth-order equation

$$A_h \nabla^4 \psi_M - \beta \frac{\partial \psi_M}{\partial x} = \nabla \times \bar{\tau}. \quad (9)$$

- If Munk's equation is used with the simplified wind stress

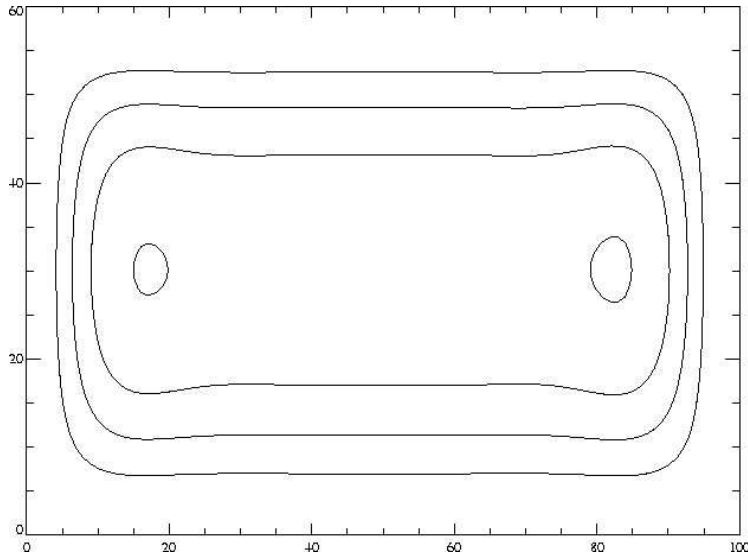
$$\tau_x = -\tau_0 \cos\left(\frac{\pi y}{y_n}\right)$$

$$\tau_y = 0$$

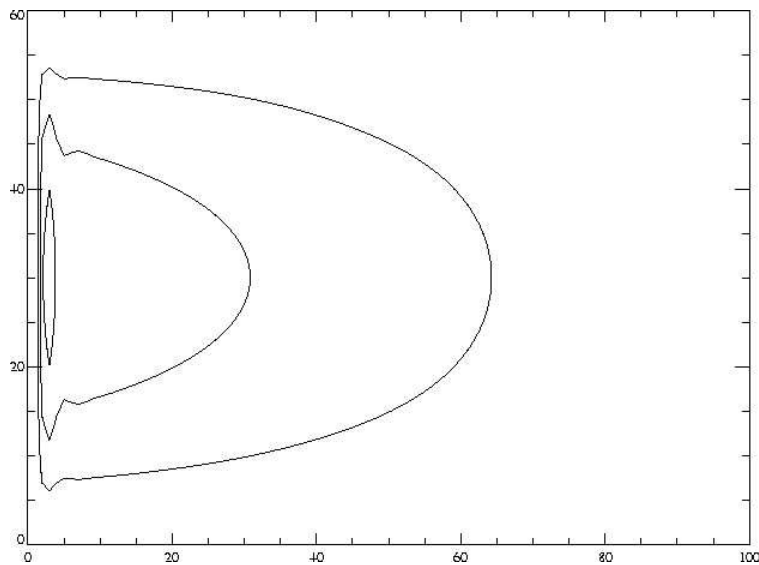
then the final equation becomes

$$A_h \nabla^4 \psi_M - \beta \frac{\partial \psi_M}{\partial x} = -\frac{\pi \tau_0}{y_n} \sin\left(\frac{\pi y}{y_n}\right). \quad (10)$$

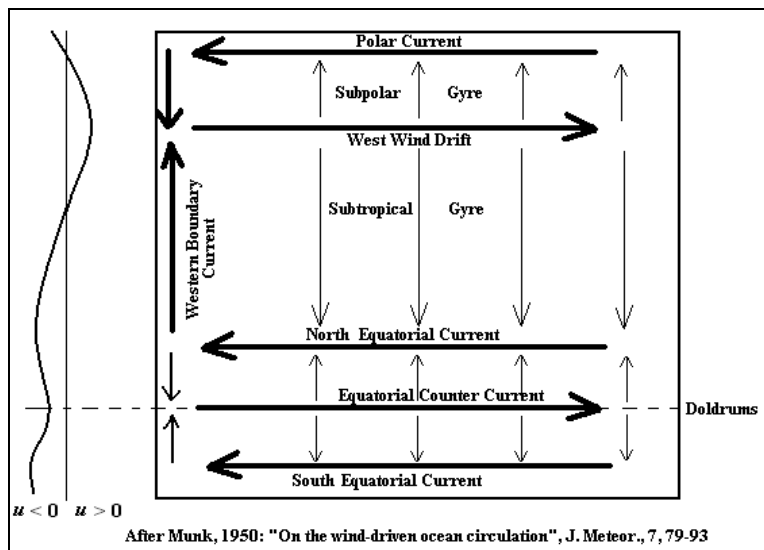
- If this equation is solved on the  $f$ -plane ( $\beta = 0$ ) then the results (shown below) are similar to that of Stommel, showing a symmetric gyre.



- On the  $\beta$ -plane the solution for the streamfunction is slightly different from Stommel's, but still shows the westward intensification of the boundary current.



- Munk actually used observed wind stresses instead of the simplified wind stress, but his results were very similar to those above. He also solved for the case of a triangular ocean basin, which more realistically represents the Atlantic Ocean.
- Munk's results show an interesting feature in that there is a southward flowing *countercurrent* just to the east of the western boundary current. This countercurrent is indeed a real feature that is observed east of the Gulf Stream.
- The idealized currents for a rectangular ocean basin are shown below.



- The major features of the circulation are:

- **A cyclonically rotating subpolar gyre**
- **An anticyclonically rotating subtropical gyre**
- **Two westward flowing equatorial currents symmetric with the ITCZ (not the Equator).**
- **An eastward flowing equatorial counter current between the equatorial currents.**
- **Strong, western boundary currents in the subtropical and subpolar gyres, contrasted with weaker return flows east of the gyre centers.**