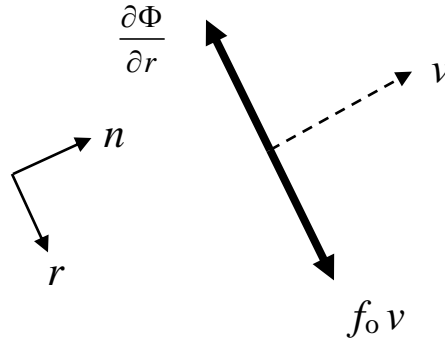


**ESCI 343 – Atmospheric Dynamics II**  
**Lesson 14 – Inertial/slantwise Instability**

Reference: *An Introduction to Dynamic Meteorology (3<sup>rd</sup> edition)*, J.R. Holton  
*Atmosphere-Ocean Dynamics*, A.E. Gill  
*Mesoscale Meteorology in Midlatitudes*, P. Markowski and Y. Richardson

**INERTIAL INSTABILITY**

Imagine an air parcel in geostrophic balance at speed  $v$  on the  $f$ -plane. The diagram below shows the balance of accelerations in the lateral direction. Note that the direction of  $v$  is completely arbitrary. The coordinate  $r$  is directed to the right of the wind, while the coordinate  $n$  is in the direction of the geostrophic wind. The wind component in the transverse direction (along  $r$ ) is denoted as  $u$ .



The momentum equations in this coordinate system are

$$\frac{Du}{Dt} = -\frac{\partial\Phi}{\partial r} + f_0 v \quad (1)$$

$$\frac{Dv}{Dt} = -f_0 u \quad (2)$$

Imagine that an air parcel starts in geostrophic balance. If the parcel is suddenly impelled laterally in the direction of  $r$  at a speed  $u$ , the balance of accelerations will change. Taking the time derivative of (1) gives us an equation for how the lateral acceleration changes with time,

$$\frac{D}{Dt} \left( \frac{Du}{Dt} \right) = -\frac{D}{Dt} \left( \frac{\partial\Phi}{\partial r} \right) + \frac{D}{Dt} (f_0 v) \quad (3)$$

The terms on the right-hand side of (3) are evaluated as follows:

$$\frac{D}{Dt} \left( \frac{\partial\Phi}{\partial r} \right) = \frac{\partial\Phi}{\partial t} + u \frac{\partial}{\partial r} \frac{\partial\Phi}{\partial r} + v \frac{\partial}{\partial n} \frac{\partial\Phi}{\partial r} = u \frac{\partial^2\Phi}{\partial r^2} \quad (4)$$

and

$$\frac{D}{Dt} (f_0 v) = f_0 \frac{Dv}{Dt} = f_0 (-f_0 u) = -f_0^2 u \quad (5)$$

so that (3) becomes

$$\frac{D^2 u}{Dt^2} = -u \frac{\partial^2 \Phi}{\partial r^2} - f_0^2 u \quad (6)$$

or

$$\frac{D^2 u}{Dt^2} + \left( \frac{\partial^2 \Phi}{\partial r^2} + f_0^2 \right) u = 0 . \quad (7)$$

The solutions to (7) will be oscillatory provided that

$$\frac{\partial^2 \Phi}{\partial r^2} + f_0^2 > 0 . \quad (8)$$

In this case, the parcel will oscillate around its original line of motion, and the flow is *inertially stable*. The angular frequency of the oscillations is

$$\omega^2 = \frac{\partial^2 \Phi}{\partial r^2} + f_0^2 . \quad (9)$$

If instead,

$$\frac{\partial^2 \Phi}{\partial r^2} + f_0^2 < 0 , \quad (10)$$

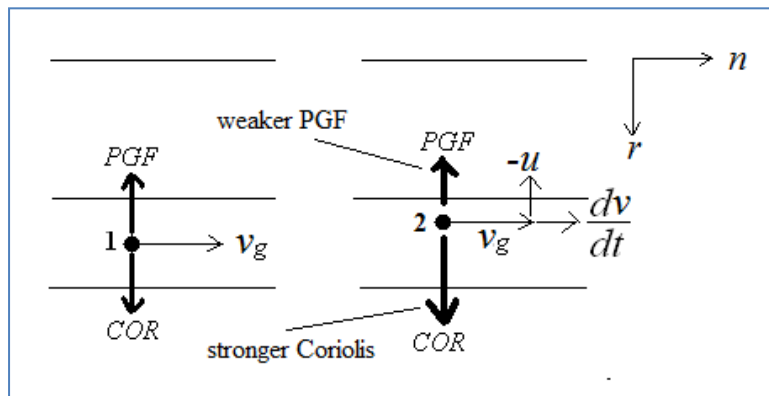
then the transverse velocity will grow exponentially with time and the parcel will accelerate away from its original line of motion.

### PHYSICAL INTERPRETATION OF INERTIAL STABILITY

The physical interpretation of inertial stability/instability is directly linked to how the pressure gradient tightens or loosens in the direction of  $r$ . The figure below shows the case of the pressure gradient becoming tighter with increasing  $r$ , which implies that

$$\frac{\partial^2 \Phi}{\partial r^2} > 0 . \quad (11)$$

The stability criteria (8) tells us that this case is inertially stable, so that a parcel displaced latitudinally will return to its base latitude. To see why this occurs, refer to the diagram below.

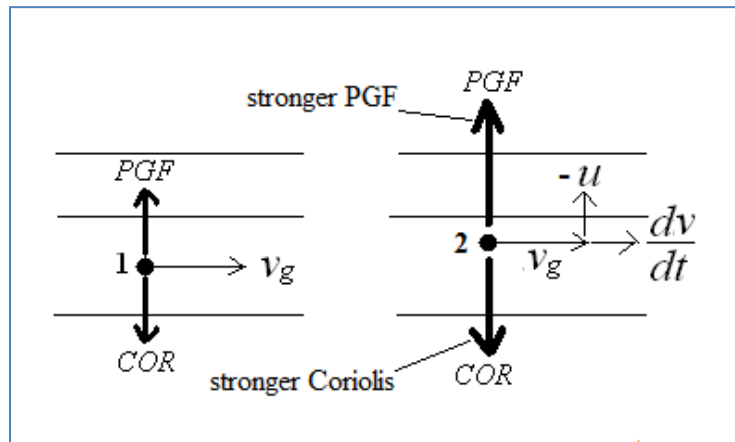


Imagine parcel in geostrophic balance at Point 1. If the parcel is perturbed in the direction of  $-r$ , then there will also be a positive acceleration in the direction of  $n$  due to Coriolis. This will increase the  $v$  component of the wind and thus increase the

component of the Coriolis acceleration in the direction of  $r$ . Since the parcel is also moving into an area of weaker pressure gradient, there is a net acceleration on the parcel toward positive  $r$ . Thus, a lateral perturbation will result in a restoring acceleration back toward the original line of motion.

For the case where the pressure gradient decreases in the  $r$  direction the physical interpretation is a little more complex. The diagram below shows this case, where

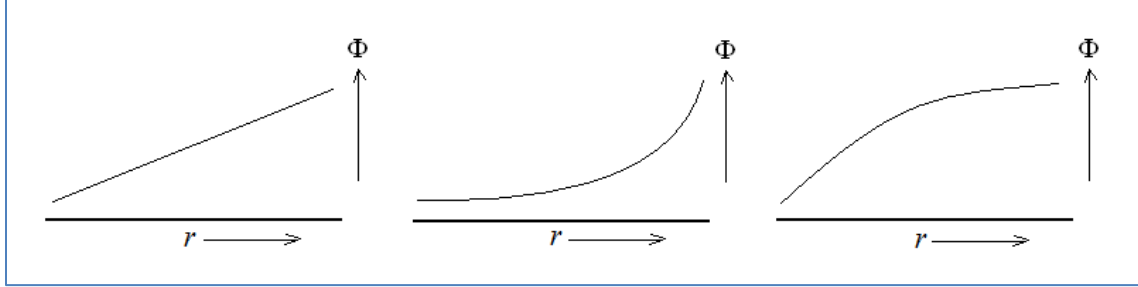
$$\frac{\partial^2 \Phi}{\partial r^2} < 0. \quad (12)$$



As before the parcel is perturbed in the negative  $r$  direction at velocity  $u$ . There is still an acceleration due to Coriolis in the  $n$  direction, which will increase the component of the Coriolis acceleration in the  $r$  direction. However, the pressure gradient acceleration is also increasing as the parcel moves to Point 2. If the pressure gradient acceleration is larger than the Coriolis acceleration, the parcel will accelerate toward the negative  $r$  direction. If, however, the increase in Coriolis acceleration outweighs the increase in the pressure gradient acceleration, the parcel will accelerate back toward its original line of motion. Thus, a decreasing pressure gradient with increasing  $r$  is not sufficient to produce inertial instability. In order for instability to occur in this case, (8) shows us that

$$\frac{\partial^2 \Phi}{\partial r^2} > -f_0^2. \quad (13)$$

Plots of geopotential versus  $r$  for a constant pressure surface are shown in the diagrams below. For the first two diagrams the atmosphere is inertially stable, because the second derivative of  $\Phi$  is either positive or zero. For the third diagram the second derivative of  $\Phi$  is negative, but instability would depend on just how negative the second derivative is.



## ABSOLUTE MOMENTUM AND INERTIAL STABILITY

The stability criteria (8) can be written in alternate forms as follows:

$$f_0^2 + \frac{\partial^2 \Phi}{\partial r^2} = f_0^2 + \frac{\partial}{\partial r} \left( \frac{\partial \Phi}{\partial r} \right) = f_0^2 + \frac{\partial}{\partial r} (f_0 v_g) > 0$$

or

$$f_0 + \frac{\partial v_g}{\partial r} > 0. \quad (14)$$

For ease of notation we can rewrite (14) as

$$\frac{\partial}{\partial r} (f_0 r + v_g) > 0,$$

and defining a quantity called the *absolute momentum*<sup>1</sup> as

$$M \equiv f_0 r + v_g \quad (15)$$

the condition for inertial stability/instability can be written as

$$\frac{\partial M}{\partial r} > 0: \text{ Inertially stable}$$

$$\frac{\partial M}{\partial r} = 0: \text{ Inertially neutral} \quad (16)$$

$$\frac{\partial M}{\partial r} < 0: \text{ Inertially unstable}$$

Even though  $M$  is called absolute momentum, it is not exactly equal to the momentum as viewed from space, but is equal to it within some function of  $r$ . The reason it is defined this way is so that its  $r$ -derivative is equal to the absolute vorticity via

$$\eta = \frac{\partial M}{\partial r}. \quad (17)$$

Thus, we can view inertial instability of the flow as occurring if the absolute vorticity of is negative. This partially explains why we don't see negative absolute vorticity occurring on the synoptic scale, because if it does occur, inertial instability will occur.

<sup>1</sup> Our derivation of inertial instability and absolute momentum was done in coordinates that are completely arbitrary, with no preferred direction for the geostrophic wind. Many basic treatments of this topic do the derivation for a purely zonal geostrophic flow. In this case, absolute momentum is defined instead as  $M = f_0 y - u_g$ , and the derivatives in (16) are taken with respect to  $y$  instead of  $r$ .

## INERTIAL STABILITY OF A VORTEX

The concept of inertial instability can be extended to curved flow (for a detailed derivation see [Lesson 10 of the Tropical Meteorology class notes](#)). In this case the background flow is assumed to be in gradient wind balance. Also, it is the radial gradient of the absolute *angular* momentum that is important, rather than the gradient of the absolute momentum. The absolute angular momentum is given by

$$M_a = vr + f_0 r^2 / 2 \quad (18)$$

where  $v$  is the tangential velocity of the vortex, and  $r$  is the distance from the vortex center. The condition for inertial stability/instability of a vortex is

$$\frac{\partial M_a^2}{\partial r} > 0: \text{ Inertially stable}$$

$$\frac{\partial M_a^2}{\partial r} = 0: \text{ Inertially neutral}$$

$$\frac{\partial M_a^2}{\partial r} < 0: \text{ Inertially unstable}$$

The relationship between absolute angular momentum and absolute vorticity is

$$\eta = \frac{1}{r} \frac{\partial M_a}{\partial r}. \quad (19)$$

Most treatments of inertial stability mention that negative absolute vorticity is inertially unstable. While this is true for weak or straight-line flow, in strong vortices it is possible to have negative absolute vorticity and still be inertially stable.

## SLANTWISE/SYMMETRIC INSTABILITY

Static stability refers to an air parcel's resistance to vertical displacement, whereas inertial instability refers to its resistance to transverse-horizontal displacement. It turns out that it is possible for a parcel to be both statically (vertically) and inertially (horizontally) stable, and yet be unstable with respect to diagonal displacement. Such instability is called slantwise instability, or symmetric instability, and may be an important instability mechanism near fronts or other baroclinic zones.

Mathematically, the condition for static instability is

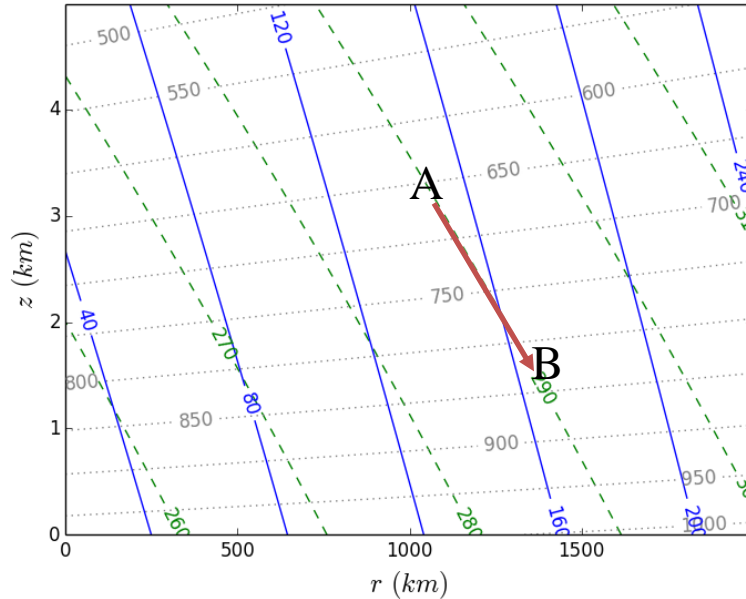
$$\left( \frac{\partial M}{\partial r} \right)_\theta > 0: \text{ Slantwise stable}$$

$$\left( \frac{\partial M}{\partial r} \right)_\theta = 0: \text{ Slantwise neutral}$$

$$\left( \frac{\partial M}{\partial r} \right)_\theta < 0: \text{ Slantwise unstable}$$

This looks just like the condition for inertial stability except the derivative is taken along an adiabatic surface, rather than on a horizontal surface.

The figure below shows lines of potential temperature (dashed) and the lines of absolute momentum (solid). The pressure surfaces are also shown with dotted lines. In this configuration the atmosphere is stable with respect to horizontal motion (inertial stability), vertical motion (static stability) and diagonal motion along an adiabat (slantwise stability). A parcel moving adiabatically from Point A to Point B would be moving into a region of higher absolute momentum, so that  $(\partial M / \partial r)_\theta > 0$  and the atmosphere is slantwise stable.



The next figure (below) shows what can happen near a baroclinic zone (front). In this case there are regions where the adiabats are more steeply sloped than the absolute momentum lines. In this region a parcel moving in the positive  $r$  direction on an adiabat from Point A to Point B would be moving into a region of lower absolute momentum, so that  $(\partial M / \partial r)_\theta < 0$  and the atmosphere is slantwise unstable.

