ESCI 343 – Atmospheric Dynamics II
Lesson 13 – Geostrophic/gradient Adjustment

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.6

GEOSTROPHIC ADJUSTMENT OF A BAROTROPIC FLUID

The atmosphere is nearly always close to geostrophic and hydrostatic balance. If this balance is disturbed through such processes as heating or cooling, the atmosphere adjusts itself to get back into balance. This process is called *geostrophic adjustment*, although it may more accurately be referred to as gradient adjustment, since in curved flow the atmosphere tends toward gradient balance. One method of studying geostrophic adjustment is to first study adjustment in a barotropic fluid using the shallow-water equations. Once we understand adjustment in a barotropic fluid we can easily extend our results to a baroclinic fluid by use of the concept of equivalent depth, studied in a previous lesson.

For this we can use the linearized shallow-water equations with zero mean flow,

$$\frac{\partial u'}{\partial t} - fv' = -g\frac{\partial h'}{\partial x} \tag{1}$$

$$\frac{\partial v'}{\partial t} + fu' = -g\frac{\partial h'}{\partial y} \tag{2}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{3}$$

If we take $\partial/\partial x$ of (1) and add it to $\partial/\partial y$ (2) we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) - f\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) = -g\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right). \tag{4}$$

Rearranging (3) we get

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = -\frac{1}{H}\frac{\partial h'}{\partial t}. \tag{5}$$

From the definition of vorticity we know that

$$\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \equiv \zeta'. \tag{6}$$

Putting (5) and (6) into (4) we get

$$\frac{\partial^2 h'}{\partial t^2} - gH\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + fH\zeta' = 0\,. \tag{7}$$

We need one more equation that relates h' and ζ'. This is the shallow-water vorticity equation, found by taking $\partial/\partial x$ of (2) and subtracting $\partial/\partial y$ (1) to get

$$\frac{\partial \zeta'}{\partial t} = -f\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right), \tag{8}$$

which, using (5), can be written (after some rearranging) as

$$\frac{\partial}{\partial t}\left(\frac{\zeta'}{f} - \frac{h'}{H}\right) = 0\,. \tag{9}$$

Integrating (9) with respect to time gives

$$\frac{\zeta'}{f} - \frac{h'}{H} = \frac{\zeta_0'}{f} - \frac{h_0'}{H}, \tag{10}$$

where ζ_0' and h_0' refers to the initial values of relative vorticity and height perturbation. Using this in (7) results in

$$\frac{\partial^2 h'}{\partial t^2} - gH\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f^2 h' = -f\left(H\zeta_0' - fh_0'\right). \tag{11}$$

Since the quantity gH is the square of the speed of a gravity wave in this fluid, we can denote it by c^2 and write this equation as

$$\frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f^2 h' = -f\left(H\zeta_0' - fh_0'\right). \tag{12}$$

Equation (12) governs the geostrophic adjustment process in a barotropic fluid.

THE STEADY-STATE SOLUTION

Lets simplify things somewhat by assuming the initial state is at rest, and has an abrupt step in the surface height given by

$$h_0' = -\hat{h}\,\mathrm{sgn}(x)\,.^{1} \tag{13}$$

We also assume that there is no dependence in the *y*-direction. Equation (12) then becomes

$$\frac{\partial^2 h'}{\partial t^2} - c^2\frac{\partial^2 h'}{\partial x^2} + f^2 h' = -f^2\hat{h}\,\mathrm{sgn}(x)\,, \tag{14}$$

[1] The sgn(x) function is defined to be +1 for $x \geq 0$, and −1 for $x < 0$.

a second order, non-homogeneous partial differential equation. The homogeneous form of this equation supports shallow-water inertial-gravity waves (see exercises). After these waves have subsided, there will remain a steady-state solution which obeys the steady state equation

$$\frac{d^2h'}{dx^2} - \left(\frac{f}{c}\right)^2 h' = \left(\frac{f}{c}\right)^2 \hat{h}\,\mathrm{sgn}(x)\,. \tag{15}$$

Equation (15) is a second-order, non-homogeneous ordinary differential equation with constant coefficients (assuming f and c are constant). The solution to (15) consists of a complementary solution (the general solution to the homogeneous equation) plus a particular solution,

$$h'(x) = h'_c(x) + h'_p(x)\,. \tag{16}$$

The complementary solution is found from the characteristic equation for the homogeneous form of (15), which is

$$r^2 - (f/c)^2 = 0\,. \tag{17}$$

Therefore, the complementary solution is then

$$h'_c(x) = Ae^{\alpha x} + Be^{-\alpha x} \tag{18}$$

where

$$\alpha \equiv f/c\,. \tag{19}$$

For the particular solution we use the ***method of undetermined coefficients***, guessing that the particular solution will have the form

$$h'_p(x) = C\,\mathrm{sgn}(x)\,. \tag{20}$$

Putting (20) into (15) shows that $C = -\hat{h}$, so that the particular solution is

$$h'_p(x) = -\hat{h}\,\mathrm{sgn}(x)\,. \tag{21}$$

Therefore, the general solution of (15) is

$$h'(x) = Ae^{\alpha x} + Be^{-\alpha x} - \hat{h}\,\mathrm{sgn}(x)\,, \tag{22}$$

or

$$h'(x) = \begin{Bmatrix} Ae^{\alpha x} + Be^{-\alpha x} - \hat{h} & x \ge 0 \\ Ae^{\alpha x} + Be^{-\alpha x} + \hat{h} & x < 0 \end{Bmatrix}. \tag{23}$$

All that remains is to apply the boundary conditions, which require that:

- $h'(x)$ remain bounded as $x \to \pm\infty$: This requires that A = 0 for positive x and B = 0 for negative x, so that

$$h'(x) = \begin{Bmatrix} Be^{-\alpha x} - \hat{h} & x \ge 0 \\ Ae^{\alpha x} + \hat{h} & x < 0 \end{Bmatrix} \tag{24}$$

- $h'(x)$ be continuous at x = 0: This means that

$$B - \hat{h} = A + \hat{h} \tag{25}$$

- The first derivative of $h'(x)$ be continuous at x = 0: This means that

$$-\alpha B = \alpha A \tag{26}$$

Solving (25) and (26) for A and B yields

$$A = -\hat{h}$$

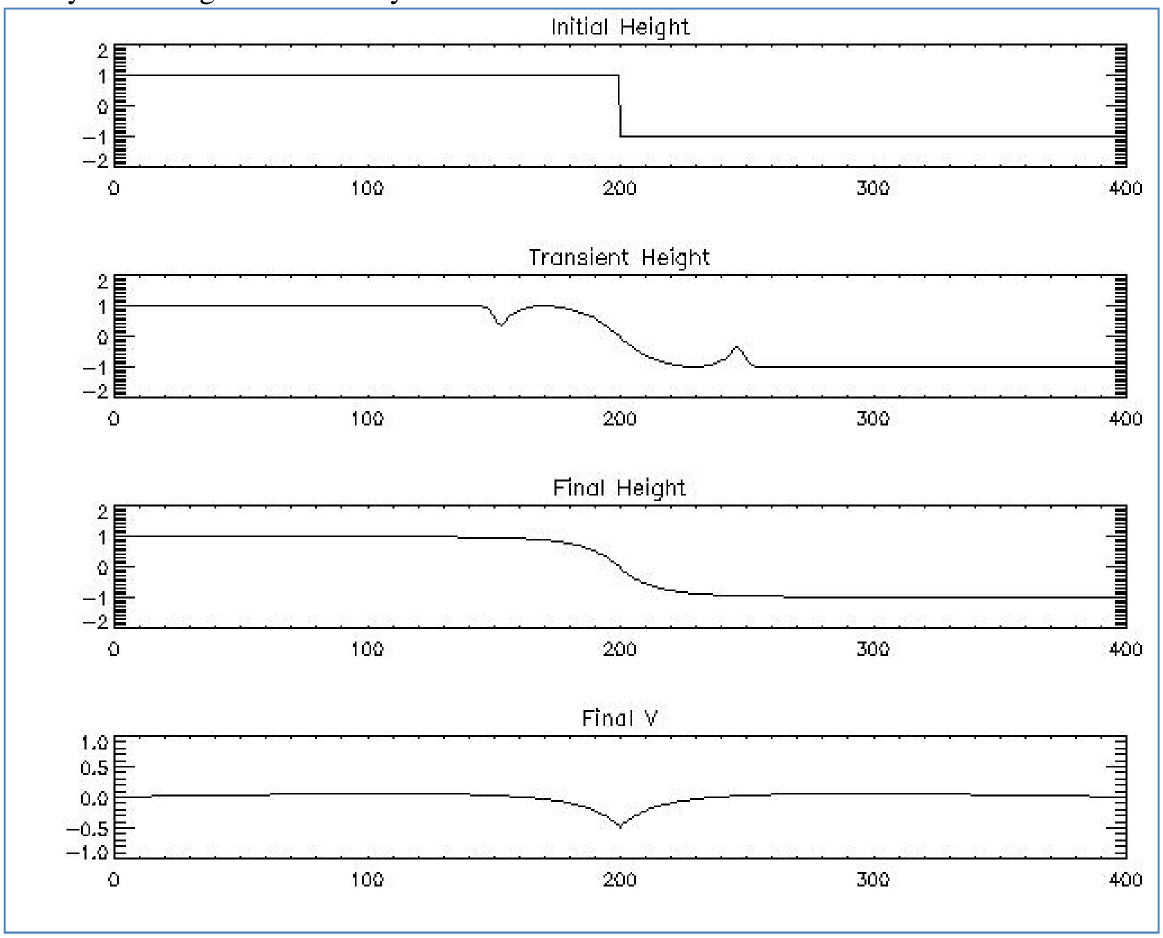
$$B = \hat{h}$$

so that the steady-state solution is

$$h'(x) = \hat{h}\begin{Bmatrix} e^{-\alpha x} - 1 & x \geq 0 \\ 1 - e^{\alpha x} & x < 0 \end{Bmatrix}. \tag{27}$$

ANALYSIS OF THE SOLUTION

The figures below show the initial height field, the transient height field, and the steady-state height and velocity fields taken from a 1-D shallow-water numerical model.



The transient solution consists of the shallow-water inertial gravity waves. The final height solution is the steady state solution from (27). The figures are striking in that, though the step that was in the initial conditions is smoothed out, there is still a region near the center of the domain with a horizontal pressure gradient, and therefore, with a geostrophic flow out of the page. The initial height field adjusted under the influence of gravity, and set up a flow that is in geostrophic balance with the final height field. The excess mass and potential energy were removed by the inertial-gravity waves which propagated away as part of the non-steady state solution.

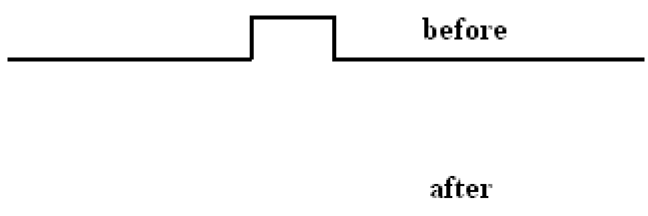
The region in which there is a remaining height gradient is characterized by an *e*-folding scale of $1/\alpha$. This length scale is of fundamental importance. It measures the scale over which the influence of the earth's rotation affects the flow, and is called the ***Rossby radius of deformation***. It is defined as

$$\lambda_R \equiv c/f \; . \tag{28}$$

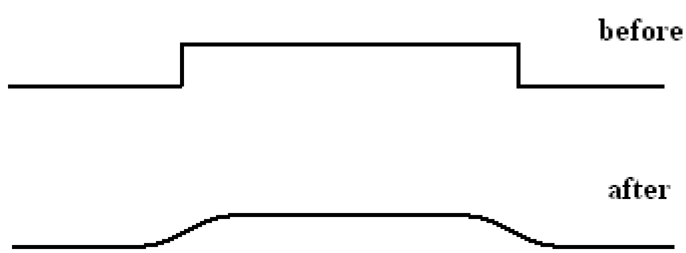
The radius of deformation is given by the group velocity of a gravity wave divided by the Coriolis parameter. The physical essence of the radius of deformation can be seen by recalling that the *inertial period* (the time scale for which rotational effects are important) is $2\pi f^{-1}$, so that $2\pi\lambda_R$ can be interpreted as the distance traveled by a gravity wave during one inertial period. For dispersive gravity wave modes the group velocity, c_g, should be used rather than the phase speed.

ROSSBY RADIUS OF DEFORMATION

The Rossby radius of deformation is a fundamental physical parameter of a fluid on a rotating reference frame. It gives a length scale that can be used as a measure of how large a disturbance has to be in order for rotational effects to be important. The physical concept of the radius of deformation is better illustrated using the following example. Imagine that you immerse a tumbler into a lake, turn it upside down, and lift it up to the point just before the lip breaks the surface of the water (a quick calculation will show that the radius of the tumbler is much smaller than the radius of deformation.) Right as you lift the lip of the tumbler completely from the water the initial height field would look like that pictured below. As soon as you lift the tumbler from the water, the water surface begins to adjust by generating gravity waves which propagate away from the initial disturbance. In the steady state all that remains after the water calms is a flat surface, as shown.

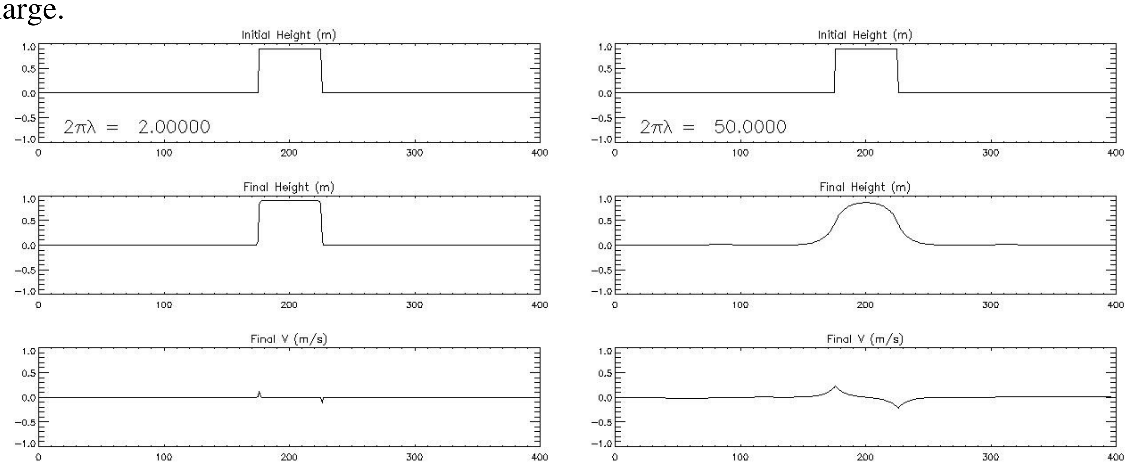


Now, imagine the same experiment performed in the ocean, only using an extremely wide tumbler that has a radius much greater than the radius of deformation. Performing the same experiment will result in a steady state solution pictured below, with a hump of water remaining, around which an anticyclonic geostrophic (actually, gradient) circulation has developed!

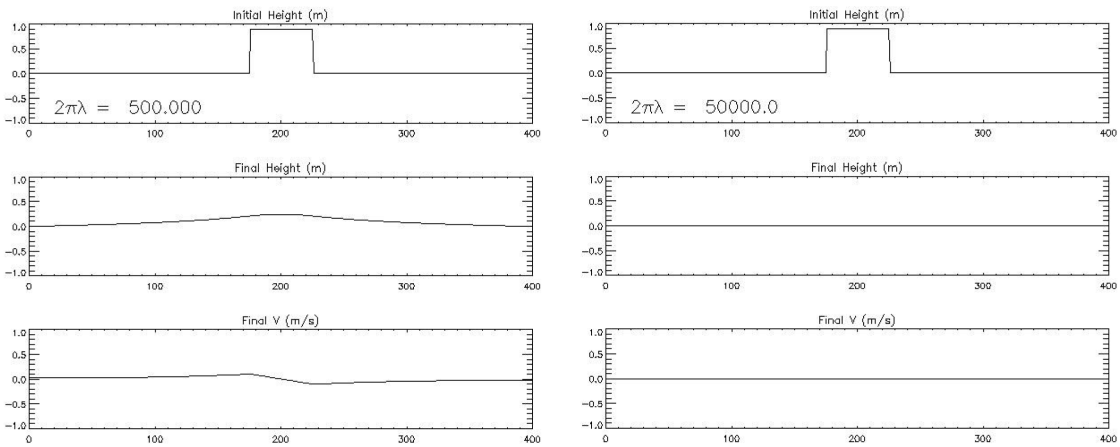


What is the difference between the two experiments? It is how the horizontal scale of the initial disturbance compares with the Rossby radius of deformation!

We can illustrate this using a 1-D shallow-water model for disturbances ranging in size from very small compared to the Rossby radius of deformation to those that are very large.



(a) (b)



(c) (d)

In these figures, the ratio of the horizontal length of the disturbance, *L* divided by $2\pi\lambda_R$ is (a) 25; (b) 1.0; (c) 0.1; and (d) 0.001 (in the model the physical size of the disturbances is the same in each case, but the Coriolis acceleration is varied to achieve a variable radius of deformation). In all cases the final height and velocity fields are in geostrophic balance. However, for very small disturbances the height field adjusted to the initial velocity field, while for very large disturbances the velocity field adjusted to the initial height field.

The following rules apply in all cases, and are suggested for commitment to memory:

If the size of the disturbance is much greater than 2π times the Rossby radius of deformation ($L >> 2\pi\lambda_R$), then the velocity field adjusts to the initial height (mass) field.[2]

[2] The terms "height" field and "mass" field are synonymous.

If the disturbance is much less than 2π times the Rossby radius of deformation ($L \ll 2\pi\lambda_R$), then the height field adjusts to the initial velocity field.

If the disturbance is of the same order as 2π times the Rossby radius of deformation ($L \sim 2\pi\lambda_R$), then the height and velocity fields undergo mutual adjustment.

In all cases, the final height and velocity fields are in geostrophic/gradient balance!

ADJUSTMENT IN A VORTEX

Our derivation of the Rossby radius of deformation was for a 1-D fluid, so there was no curvature to the flow. The general form of the Rossby radius of deformation for a vortex is

$$\lambda_R = \frac{c}{\sqrt{\eta\left(f_0 + 2v/r\right)}} \tag{29}$$

where η is the absolute vorticity, v is the tangential velocity, and r is the radius of curvature of the flow (see the notes for Tropical Meteorology, Lesson 9, if you are interested in a derivation for a vortex). For flows whose absolute vorticity is primarily due to the earths vorticity (i.e., flows where $\zeta \ll f$) this becomes

$$\lambda_R = c/f\ ,$$

which is the same as what we derived here.

GEOSTROPHIC ADJUSTMENT IN A MULTI-LAYER FLUID

The principles of geostrophic adjustment in a multi-layer, hydrostatic fluid are identical to that in a barotropic fluid, except that there are several modes of inertial-gravity waves generated: one for the barotropic mode, and one for each baroclinic mode. Each mode has its own unique radius of deformation, given by

$$\begin{aligned} \lambda_0 &= c_0/f \\ \lambda_1 &= c_1/f \\ &\vdots \\ \lambda_n &= c_n/f \end{aligned} \tag{30}$$

where the subscript 0 refers to the barotropic mode, and the subscript n refers to the n^{th} baroclinic mode. The barotropic mode is often called the *external mode*, and its radius of deformation the *external radius of deformation*. In a two-layer fluid the baroclinic mode is often called the *internal mode*, and its radius of deformation the *internal radius of deformation*. In a continuously stratified fluid (such as the ocean or atmosphere) there are in theory an infinite number of baroclinic modes possible; however, most of the energy is confined to a few of the lower baroclinic modes, so application of geostrophic adjustment is greatly simplified, as we can concern ourselves with a smaller, finite number of modes.

In a continuously stratified fluid the group velocity of the modes of oscillation can be approximated as

$$c_n \cong NH/n\pi; \quad n = 0,1,2,\cdots \tag{31}$$

where N is the Brunt-Vaisala frequency and H is the scale height. In this case the Rossby radii of deformation are

$$\lambda_n \cong \frac{NH}{nf\pi}; \quad n = 0,1,2,\cdots. \tag{32}$$

Since the baroclinic modes have a much smaller wave speed than the barotropic mode, the baroclinic radius of deformation is much smaller than that of the barotropic radius of deformation (see exercises).

SUMMARY AND FURTHER DISCUSSION

On the synoptic scale, the atmosphere is close to geostrophic and hydrostatic balance. Radiational heating and cooling, latent heat release, and other factors push the atmosphere from geostrophic balance. The atmosphere adjusts back into geostrophic balance by generating inertial-gravity waves which propagate energy away from the disturbance. The nature of the adjustment depends on how the horizontal scale of the disturbance compares with the Rossby radius of deformation. Since the atmosphere is stratified, it is usually the baroclinic radii of deformation that are important.

For small-scale phenomena, such as individual thunderstorms, the disturbance is much smaller than the radius of deformation. Therefore, the mass field adjusts to the initial velocity field, and no residual synoptic scale circulations are generated. However, larger-scale phenomena can be of the order of the baroclinic radius of deformation, and can therefore leave a synoptic scale circulation as the velocity field adjusts to the mass field.

The radius of deformation also provides us with a length scale by which to gauge whether phenomena are effected by the earth's rotation. Phenomenon whose horizontal length scales are much smaller than the radius of deformation are unlikely to be effected by the earth's rotation (unless they persist for a time scale on the order of the inertial period, $2\pi f^{-1}$. Therefore, thunderstorms, tornadoes, and toilets are not affected by the earth's rotation.

EXERCISES

1. Show that the homogeneous form of equation (12),

$$\frac{\partial^2 h'}{\partial t^2} - c^2 \frac{\partial^2 h'}{\partial x^2} + f^2 h' = 0,$$

supports shallow-water inertial-gravity waves having a dispersion relation of

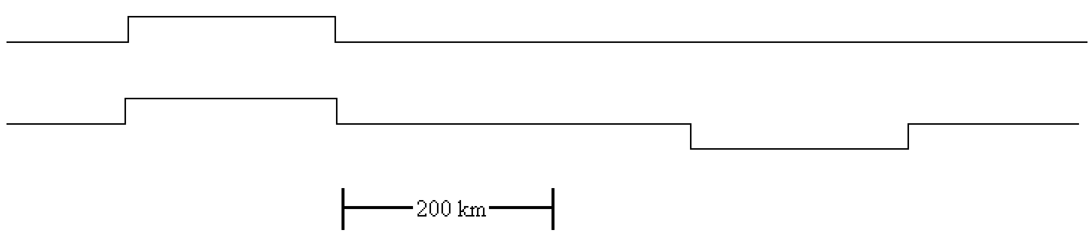
$$\omega^2 = f^2 + c^2 k^2.$$

2. The ocean is often represented as a two-layer fluid. Assume the upper layer has a depth of 700 m and a density of 1021 kg/m^3, while the lower layer has a depth of 3300 m and a density of 1023 kg/m^3.

a. Find the barotropic (external) radius of deformation at latitude 45°N.

b. Find the baroclinic (internal) radius of deformation at the same latitude.

c. For the disturbances in this ocean shown below, sketch the final position of the upper and lower surfaces. Assume the disturbance on the left only generates waves in the barotropic mode, while the disturbance on the right only generates waves in the baroclinic mode. The top line represents the external surface, while the bottom line represents the internal interface.



3. Calculate the radius of deformation for a typical bathtub. How large would a disturbance in the tub have to be in order for rotational effects to be important?

4. Does the radius of deformation increase or decrease with latitude?