ESCI 343 – Atmospheric Dynamics II
Lesson 12 - Inertial-gravity Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.5

INERTIAL-GRAVITY WAVES

Inertial-gravity waves occur when a statically stable flow is also inertially stable. They are essentially gravity waves that have a large enough wavelength to be affected by the earth's rotation. To study inertial-gravity waves we need to include the Coriolis terms in the governing equations. For simplicity, we will use the incompressible continuity equation. Therefore, the linearized governing equations are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} + fv' \tag{1}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} - fu' \tag{2}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}}g \tag{3}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{4}$$

and

$$\rho' = \frac{\bar{\rho}}{g}N^2\Delta z\,. \tag{5}$$

Combining (3) and (5) to eliminate ρ' gives

$$\frac{\partial^2 w'}{\partial t^2} = -\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial z\partial t} - N^2 w'\,. \tag{6}$$

Equations (1), (2), (4), and (6) are the governing equations for inertial-gravity waves. Assuming the usual sinusoidal solutions

$$u' = Ae^{i(kx+ly+mz-\omega t)}$$

$$v' = Be^{i(kx+ly+mz-\omega t)}$$

$$w' = Ce^{i(kx+ly+mz-\omega t)}$$

$$p' = De^{i(kx+ly+mz-\omega t)}$$

and substituting into (1), (2), (4), and (6) results in the following algebraic equations

$$i\omega A + fB - \left(ik/\bar{\rho}\right)D = 0 \tag{7}$$

$$fA - i\omega B + \left(il/\bar{\rho}\right)D = 0 \tag{8}$$

$$kA + lB + mC = 0 \tag{9}$$

$$\left(\omega^2 - N^2\right)C - \left(m\omega/\bar{\rho}\right)D = 0 \tag{10}$$

and the resulting dispersion relation for inertial-gravity waves

$$\omega^2 = \left(f^2 m^2 + N^2 \mathrm{K}_H^2\right) \big/ \mathrm{K}^2 . \tag{11}$$

Notice that if the effects of rotation are ignored ($f = 0$) then the dispersion relation becomes that for pure internal waves.

The dispersion relation can be written in terms of the propagation angle of the waves. The wave number vector makes an angle φ with the horizontal plane, and this angle is given by

$$\varphi = \arctan\left(m / K_H\right). \tag{12}$$

Equation (11) can then be written as

$$\omega^2 = f^2 \sin^2 \varphi + N^2 \cos^2 \varphi . \tag{13}$$

From this we see that the frequency of inertial-gravity waves is constrained to always lie between f and N,

$$f \le \omega \le N . \tag{14}$$

Purely horizontally propagating waves have a frequency of N, while purely vertically propagating waves have a frequency of f.

Since inertial-gravity waves have long enough wavelengths to be effected by the earth's rotation, we can assume that they are in hydrostatic balance. This implies that $m >> \mathrm{K}_H$. Therefore, we sometimes write the dispersion relation for inertial-gravity waves as

$$\omega^2 \cong f^2 + \frac{N^2 \mathrm{K}_H^2}{m^2} . \tag{15}$$

DISPERSION AND STRUCTURE OF INERTIAL-GRAVITY WAVES

The phase velocity of inertial gravity waves is

$$\vec{c} = \pm \sqrt{f^2 + \frac{N^2 \mathrm{K}_H^2}{m^2}} \frac{\vec{\mathrm{K}}}{\mathrm{K}^2} , \tag{16}$$

while the group velocity is

$$\vec{c}_g = \pm \frac{N^2}{m^2 \sqrt{f^2 + N^2 \mathrm{K}_H^2 / m^2}} \left(k\hat{i} + l\hat{j} - \frac{\mathrm{K}_H^2}{m} \hat{k} \right). \tag{17}$$

Thus, for inertial-gravity waves the group velocity and phase velocity are orthogonal, and upward propagating waves transport energy downward, while downward propagating waves transport energy upward.

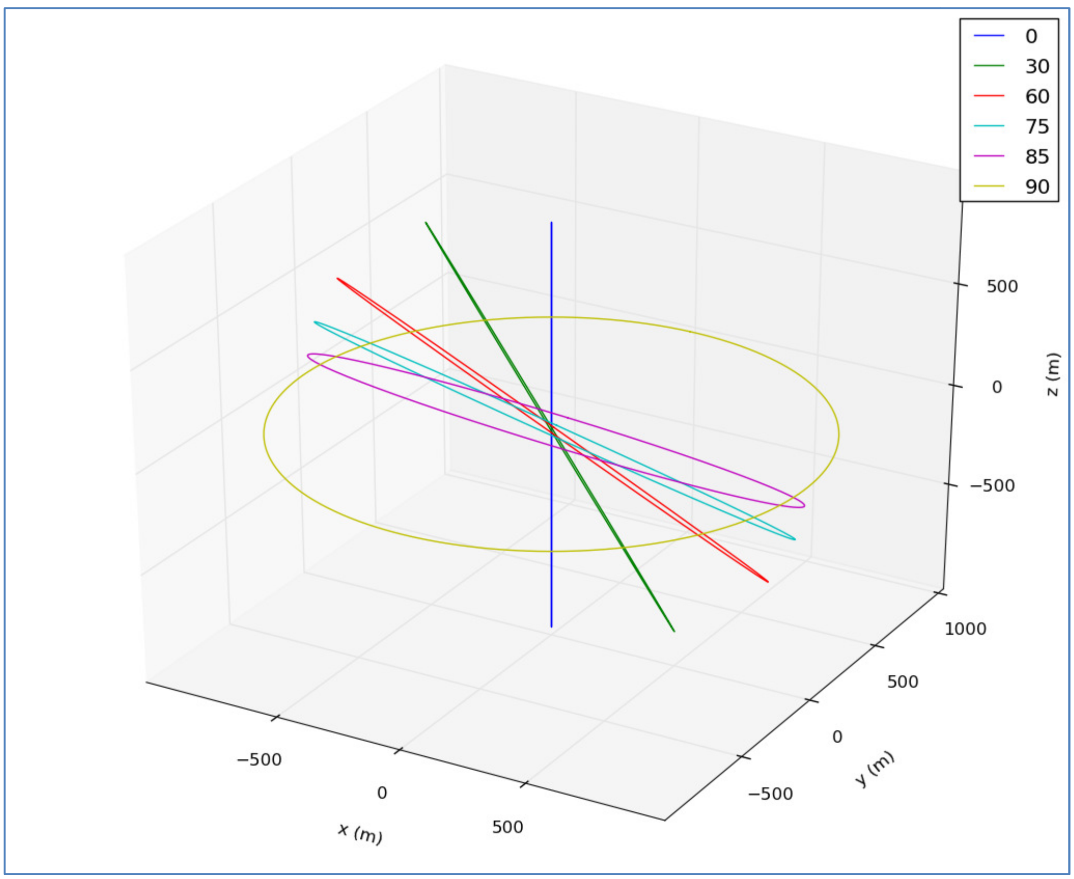
The parcel trajectories for inertial-gravity waves are ellipses. From (9) we can deduce that

$$\vec{K} \bullet \vec{V} = 0$$

(see exercises). This means that the velocity is always perpendicular to the wave number vector; i.e., there is no component of particle motion along the direction of the phase propagation. Therefore, the ellipses are always at 90° to the direction of phase propagation. The particles move along these ellipses in an anticyclonic fashion (in the

Northern Hemisphere), regardless of whether the wave propagation is up or down. This makes sense, since the Coriolis force must be directed toward the inside of the ellipse. The figure below shows the direction of the particle trajectories for upward and downward propagating waves.

The figures below show the 3-dimensional trajectories for waves traveling in the positive x and z directions for various propagation angles. Notice that the trajectory is a vertical lines for a propagation angle of zero (when $\omega = N$), as would be expected for a pure internal wave. As the propagation angle increases, ω decreases and the trajectories begin to slant, and also open up into ellipses. At the lowest frequency possible ($\omega = f$) the trajectory is a circle that lies completely in the horizontal plane.



EXERCISES

1. Show that if $m >> K_H$ then (11) becomes (15).

2. Show that the group velocity for inertial-gravity waves is given by equation (11)

3. Show that for inertial-gravity waves, $\vec{c} \bullet \vec{c}_g = 0$.

4. Use equations (7), (8), (9), and (10) to show the following phase relations between u', v', and w' for a wave traveling in the x-z plane (l = 0).

a. $u' = (i\omega/f)v'$

b. $u' = -(m/k)w'$

c. $v' = (ifm/k\omega)w'$

d. If $w' = \cos(kx + mz - \omega t)$, what are u' and v'?

5. a. Use the results from 4.a to determine whether the horizontal velocity vector will rotate cyclonically or anticyclonically with time. Will this change if the wave is propagating upward versus downward?

b. Use the results from 4.a to determine whether the horizontal velocity vector will rotate cyclonically or anticyclonically with height. Will this change if the wave is propagating upward versus downward?

6. Show that $\vec{K} \bullet \vec{V} = 0$ is the same as equation (9).