

ESCI 343 – Atmospheric Dynamics II

Lesson 10 - Topographic Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton

Reading: Holton, Section 7.4.2

STATIONARY WAVES

Waves will appear to be stationary if their phase speed is equal and opposite to the mean flow,

$$c = -\bar{u}. \quad (1)$$

Stationary waves will have a frequency of zero, since they do not oscillate in time, only in space.

For dispersive waves, the wavelength of the stationary wave will correspond to that wavelength which has a phase speed equal and opposite to the mean flow. None of the other wavelengths will be stationary.

An example of stationary waves is the stationary wave pattern that forms in a river when it flows over a submerged rock or obstacle. The flow of the obstacle generates many different waves of various wavelengths, but only the ones whose phase speed is equal and opposite to the flow will remain stationary. If the flow speeds up or slows down, the wavelength of the stationary wave will change.

DISPERSION RELATION FOR STATIONARY INTERNAL GRAVITY WAVES

In the atmosphere, flow of a stably stratified fluid over a mountain barrier can also generate standing waves. If we consider the flow over a sinusoidal pattern of ridges that are perpendicular with the x -axis, and with a horizontal wave number of k , then we can analyze the structure of these waves. Since these waves are internal gravity waves in the presence of a mean flow their phase speed and dispersion relation is simply

$$c = \bar{u} \pm \frac{N}{\sqrt{k^2 + m^2}} \quad (2)$$
$$\omega = \bar{u}k \pm \frac{kN}{\sqrt{k^2 + m^2}}$$

(remember that since these are linear waves, the mean flow simply adds a term $\bar{u}k$ to the frequency). But, we are only interested in the standing waves generated by the topography, for which the phase speed (and therefore, frequency) is zero. For standing waves then, we have

$$\bar{u} \pm \frac{N}{\sqrt{k^2 + m^2}} = 0. \quad (3)$$

The horizontal wave number, k , is determined by the wave number of the terrain; \bar{u} and N are properties of the atmosphere. Since these quantities are predetermined, then there is only one value of vertical wave number, m , which can satisfy the dispersion relation. Thus, m is given by

$$m^2 = \frac{N^2}{\bar{u}^2} - k^2. \quad (4)$$

Equation (4) is the dispersion relation for stationary internal gravity waves.

VERTICALLY PROPAGATING VERSUS VERTICALLY DECAYING WAVES

The vertical structure of the standing waves is determined by how N , k , and \bar{u} relate to one another.

Vertically propagating waves. If $k < N/\bar{u}$ then the right hand side is positive, and m is real. This corresponds to waves that propagate vertically, since the sinusoidal form for the dependent variables is

$$e^{i(kx+mz)}. \quad (5)$$

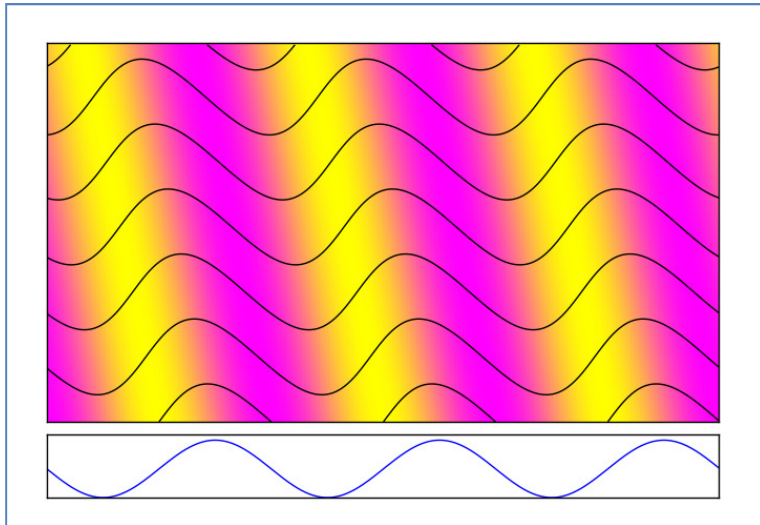
Along lines of constant phase ($kx + mz$) the following relation holds

$$kx + mz = b \quad (6)$$

where b is a constant. This means that phase lines obey the following equation,

$$z = -\frac{k}{m}x + \frac{b}{m}. \quad (7)$$

This tells us that the lines of constant phase tilt upwind with height, since they have a negative slope (see figure below). Since the wave is propagating upstream (it has to be, since it is a standing wave), the individual wave crests have a downward component of phase speed. This means that the group velocity is upward, so that topographically forced waves propagate energy upward.

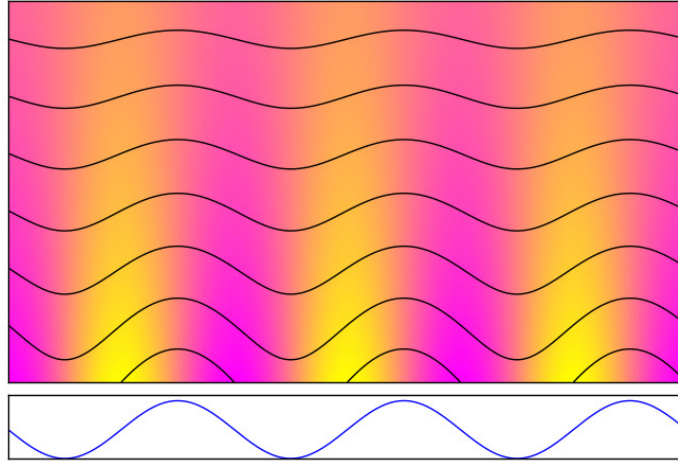


Vertically propagating topographically-forced stationary waves. The wind speed is from left-to-right. This is for $\bar{u} < N/k$, so the value of m^2 is positive and m is real. Solid lines are streamfunction; colors are for vertical velocity.

Vertically decaying waves. If $k > N/\bar{u}$ then the right-hand-side negative. In this case the sinusoidal form for the dependent variables is

$$e^{ikx} e^{-mz},$$

and the waves decay with height (such waves are also known as *evanescent*). In this case, lines of constant phase are vertical. This is illustrated in the figure below.



Vertically decaying topographically-forced stationary waves. This is for $\bar{u} > N/k$, so the value of m^2 is negative and m is imaginary. Solid lines are streamfunction; colors are for vertical velocity.

WAVES GENERATED BY AN ISOLATED MOUNTAIN RIDGE

In the real world, mountains are not pure sinusoids. However, through Fourier analysis, we can approximate the real topography by its Fourier components, with the component of wave number k having an amplitude of $H(k)$. $H(k)$ and $h(x)$ are the *Fourier transforms* of each other, and are defined by the following equations

$$h(x) = \int_{-\infty}^{\infty} H(k) e^{-ikx} dk \quad (8)$$

$$H(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x) e^{ikx} dx$$

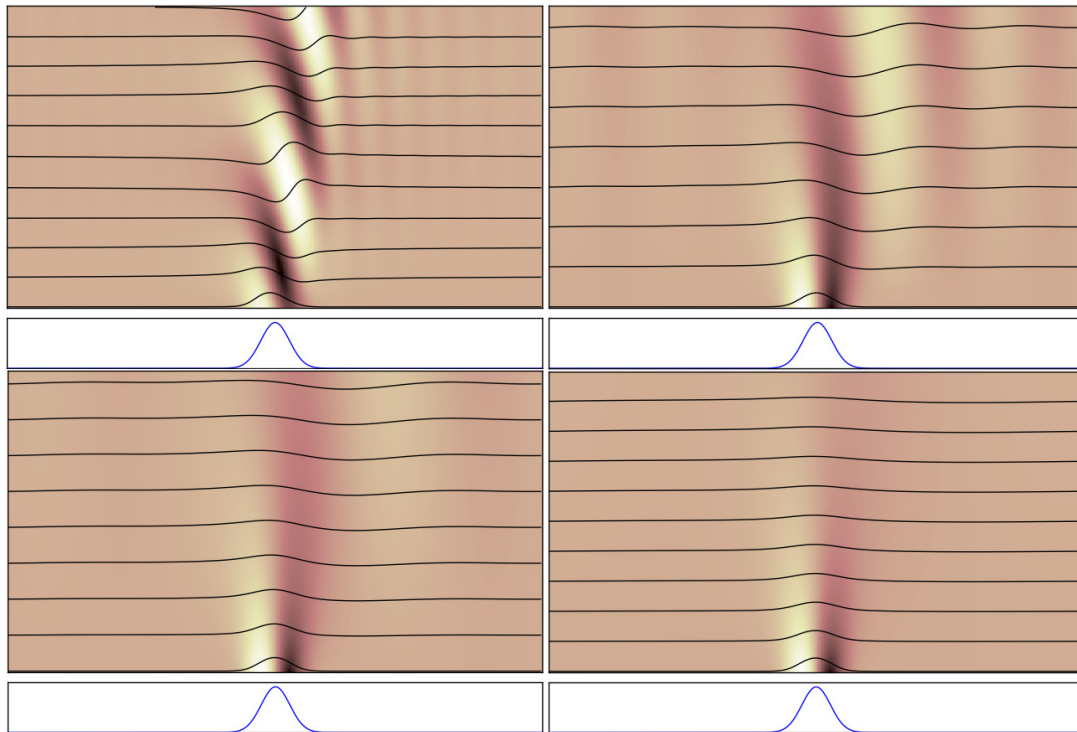
A very sharp, or discontinuous function has a greater number of high frequency (high wave number) components in its transform. A broad function has few high frequency components, and is mostly made up of low frequency (low wave number) components.

Flow over a mountain will generate a whole spectrum of gravity waves. Each wave component generated will either propagate vertically, or decay vertically, depending on whether its vertical wave number (m) determined from

$$m^2 = \frac{N^2}{\bar{u}^2} - k^2 \quad (9)$$

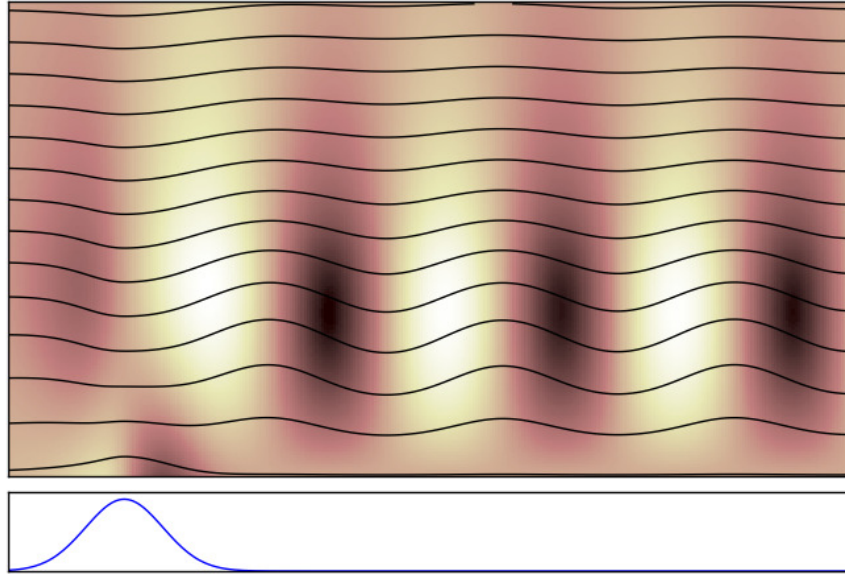
is real or imaginary. Based on the previous discussion of Fourier transforms, we expect that a narrow mountain will generate a lot of high wave number gravity waves, while a broad mountain will generate more low wave-number gravity waves.

Whether or not flow over an isolated mountain will generate vertically propagating waves, or vertically decaying waves, depends on the wind speed and the width of the mountain. Vertically propagating waves are more likely if the wind speed is slow, or the mountain is wide. Faster wind speeds and narrower mountains are more likely to result in vertically decaying waves. The plots below are for various wind speeds over a Gaussian-shaped hill. The wind speed in each plot is constant, but is faster in each successive plot.



TRAPPED (LEE) WAVES

In the previous discussion on mountain waves, we've assumed that the mean flow does not have vertical shear, and that the static stability is constant with height. Since wind speed normally increases with height it is possible that (9) will yield vertically propagating waves in the lower layer, but have vertically decaying waves in the upper part of the atmosphere. This is illustrated in the figure below. In this figure there are two distinct vertical layers. In each layer the quantity N/\bar{u} (called the *Scorer parameter*) is constant (though N and \bar{u} themselves may vary within each layer). If the upper layer has a larger wind speed and/or lower stability, then the scorer parameter is larger in the lower layer than in the upper layer. This results in waves that are 'trapped' in the lower layer downwind of the mountain.



EXERCISES

1. For an isothermal, compressible atmosphere show the following (remember that $H = R'T/g$):
 - a. $c_s^2 = \gamma gH$
 - b. $N^2 = (g/H)(1 - \gamma^{-1})$

2. Assume that the Allegheny Mountains can be approximated as a parallel series of ridges approximately 25 km apart. Also, assume an isothermal, compressible atmosphere with a scale height of 8100 m. Calculate the critical wind speed below which topographically forced waves will propagate vertically, and above which they will decay with height.