ESCI 343 – Atmospheric Dynamics II
Lesson 9 – Internal Gravity Waves

References: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
A*tmosphere-Ocean Dynamics*, A.E. Gill
Waves in Fluids, J. Lighthill

Reading: Holton, 7.4.1

THE BRUNT-VÄISÄLÄ FREQUENCY

Before progressing with an analysis of internal waves we should review the important concept of the Brunt-Väisälä frequency. The vertical acceleration on an air parcel is

$$\frac{D\tilde{w}}{Dt} = -\frac{1}{\tilde{\rho}}\frac{\partial \overline{p}}{\partial z} - g \tag{1}$$

where quantities with a '~' character are properties of the air parcel, while those with an overbar are for the surrounding environment. Assuming that the atmosphere is in hydrostatic balance we can write

$$\frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g \tag{2}$$

and so

$$\frac{D\tilde{w}}{Dt} = -\frac{1}{\tilde{\rho}}\left(-\overline{\rho} g\right) - g = -\left(\frac{\tilde{\rho} - \overline{\rho}}{\tilde{\rho}}\right) g\ . \tag{3}$$

Defining the perturbation density as the difference in density between the parcel and its surrounding air at the same level,

$$\rho' = \tilde{\rho} - \overline{\rho}\,, \tag{4}$$

then (3) can be written as

$$\frac{D\tilde{w}}{Dt} = -\frac{\rho'}{\tilde{\rho}} g\ . \tag{5}$$

If the parcel starts out at level z_0 and has the same density as its environment,

$$\tilde{\rho}(z_0) = \overline{\rho}(z_0) \tag{6}$$

and is displaced adiabatically a small vertical distance *z*, then its new density will can be expressed as a Taylor series expansion

$$\tilde{\rho}(z_0 + z) \cong \overline{\rho}(z_0) + \frac{\partial \tilde{\rho}}{\partial z} z = \overline{\rho}(z_0) + \frac{\partial \tilde{\rho}}{\partial \tilde{p}}\frac{\partial \tilde{p}}{\partial z} z\ . \tag{7}$$

As parcels rise and expand their pressure instantaneously adjusts to be equal to that of the surrounding environment, so that

$$\tilde{p} = \bar{p} \tag{8}$$

and therefore

$$\frac{\partial \tilde{p}}{\partial z} = \frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g \ . \tag{9}$$

Furthermore, we know that

$$\left(\frac{\partial \tilde{\rho}}{\partial \tilde{p}} \right)_\theta = \frac{1}{c_s^2} \ . \tag{10}$$

Using (9) and (10), (7) becomes

$$\tilde{\rho}(z_0 + z) \cong \bar{\rho}(z_0) - \frac{\bar{\rho} g}{c_s^2} z \ . \tag{11}$$

The density of the environment can also be expanded using Taylor series,

$$\bar{\rho}(z_0 + z) \cong \bar{\rho}(z_0) + \frac{\partial \bar{\rho}}{\partial z} z \ . \tag{12}$$

From (11) and (12) the perturbation density, (4), becomes

$$\rho' = -\left(\frac{\partial \bar{\rho}}{\partial z} + \frac{\bar{\rho} g}{c_s^2} \right) z \tag{13}$$

and so (5) becomes

$$\frac{D\tilde{w}}{Dt} = \frac{\left(g \dfrac{\partial \bar{\rho}}{\partial z} + \dfrac{\bar{\rho} g^2}{c_s^2} \right) z}{\tilde{\rho}} \ . \tag{14}$$

For small displacements the denominator can be approximated as

$$\tilde{\rho} \cong \bar{\rho} \tag{15}$$

and so (14) is now

$$\frac{D\tilde{w}}{Dt} = \left(\frac{g}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial z} + \frac{g^2}{c_s^2} \right) z \ . \tag{16}$$

Equation (16) has the form of

$$\frac{D^2 z}{Dt^2} + N^2 z = 0 \tag{17}$$

where

$$N^2 \equiv -\left(\frac{g}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial z} + \frac{g^2}{c_s^2} \right) . \tag{18}$$

Equation (17) is a homogeneous 2nd-order differential equation. Although N^2 is a function of *z*, in those cases were is a constant then (17) has solutions given by

$$z(t) = Ae^{\sqrt{-N^2} t} + Be^{-\sqrt{-N^2} t} \ . \tag{19}$$

These solutions are fundamentally different depending on whether N^2 is positive or negative.

If N^2 is positive, N itself if real, and solutions to (19) are

$$z(t) = Ae^{iNt} + Be^{-iNt} \tag{20}$$

which are oscillations having an angular frequency of N. N is therefore a fundamental frequency of the oscillation, and is referred to as the *Brunt-Väisälä frequency* (or buoyancy frequency).

If N^2 is negative, then N itself is imaginary, and solutions to (19) are

$$z(t) = Ae^{|N|t} + Be^{-|N|t}\,. \tag{21}$$

These solutions are exponential with time, and are not oscillatory.

The Brunt-Väisälä frequency frequency is directly related to the static stability of the atmosphere.

N^2	***N***	***Solutions for z(t)***	***Static Stability***
positive	real	oscillations	stable
negative	imaginary	exponential growth	unstable

For the atmosphere

$$\frac{g}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial z} >> \frac{g^2}{c_s^2} \tag{22}$$

and so we can get away with defining N^2 as

$$N^2 \cong -\frac{g}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial z}\,. \tag{23}$$

Also, for an ideal gas, the Brunt-Väisälä frequency can be written in terms of potential temperature as

$$N^2 = \frac{g}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial z}\,. \tag{24}$$

Equation (24) is valid for an ideal gas only, whereas (18) is true for any fluid. For an ideal gas, (18) and (24) are equivalent (see exercises).

An additional result that will be of use in the next section is that from (13) and (18) we can derive a direct relationship between the Brunt-Väisälä frequency and the perturbation density,

$$\rho' = -\frac{\bar{\rho}N^2}{g}z\,. \tag{25}$$

DISPERSION RELATION FOR PURE INTERNAL WAVES

For the present discussion we will ignore changes in density due to local compression or expansion, which is a valid assumption as long as the waves are short compared to the scale at which the density changes with height (large values of wave number). We will therefore use the linearized, anelastic continuity equation, so that the governing equations are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \tag{26}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} \tag{27}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}}g \tag{28}$$

$$w'\frac{d\bar{\rho}}{dz} = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) \tag{29}$$

which when written in *flux form* are

$$\frac{\partial}{\partial t}(\bar{\rho}\,u') = -\frac{\partial p'}{\partial x} \tag{30}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\,v') = -\frac{\partial p'}{\partial y} \tag{31}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\,w') = -\frac{\partial p'}{\partial z} - \rho' g \tag{32}$$

$$\frac{\partial}{\partial x}(\bar{\rho}\,u') + \frac{\partial}{\partial y}(\bar{\rho}v') + \frac{\partial}{\partial z}(\bar{\rho}\,w') = 0\,. \tag{33}$$

Equations (30) thru (33) are four equation in five unknowns (u', v', w', p', and ρ'). The fifth equation is found by taking $\partial^2/\partial t^2$ of (25) to get

$$\frac{\partial^2 \rho'}{\partial t^2} = \frac{\bar{\rho}}{g}N^2\frac{\partial w'}{\partial t} = \frac{N^2}{g}\frac{\partial}{\partial t}(\bar{\rho}w') \tag{34}$$

Though we could write sinusoidal solutions for the five dependent variables and solve a 5×5 determinant to get the dispersion relation, this would be tedious. We can eliminate u', v', and w' from the equations and reduce the number of equations to two as follows:

- Take $\partial/\partial t$ of (33), and combining it with $\partial/\partial x$ of (30), $\partial/\partial y$ of (31), and $\partial/\partial z$ of (32) to get

$$\frac{\partial \rho'}{\partial z} = -\frac{1}{g}\nabla^2 p'\,. \tag{35}$$

- Eliminate w' between (32) and (34) to get

$$\frac{\partial^2 \rho'}{\partial t^2} + N^2\rho' = -\frac{N^2}{g}\frac{\partial p'}{\partial z}\,. \tag{36}$$

Equations (35) and (36) are two equations in two unknowns. Substituting the sinusoidal solutions

$$p' = Ae^{i(kx++ly+mz-\omega t)}$$
$$\rho' = Be^{i(kx+ly+mz-\omega t)} \tag{37}$$

into equation set (35) and (36) yields the following dispersion relation for internal gravity waves of

$$\omega = \pm \frac{\sqrt{k^2+l^2}N}{\sqrt{k^2+l^2+m^2}} = \pm \frac{\mathrm{K}_H N}{\mathrm{K}} \tag{38}$$

where $\mathrm{K}_H = \sqrt{k^2+l^2}$ is the horizontal wave number, and $\mathrm{K} = \sqrt{k^2+l^2+m^2}$ is the total wave number.

ANALYSIS OF INTERNAL WAVE DISPERSION

The dispersion relation for internal waves (38) shows that for purely horizontal waves ($\mathrm{K} = \mathrm{K}_H$) the frequency is equal to the Brunt-Väisälä frequency. For non-horizontally traveling waves the frequency is less than the Brunt-Väisälä frequency. Therefore, the Brunt-Väisälä frequency is an upper-limiting frequency for internal waves. In other words, for internal waves $\omega^2 \le N^2$.

The phase speed for internal waves is given by

$$c = \frac{\omega}{\mathrm{K}} = \pm \frac{\mathrm{K}_H N}{\mathrm{K}^2}\,. \tag{39}$$

The phase velocity is given by

$$\vec{c} = \frac{\omega}{\mathrm{K}^2}\vec{\mathrm{K}} = \pm \frac{\mathrm{K}_H N}{\mathrm{K}^3}\left(k\hat{i} + l\hat{j} + m\hat{k}\right). \tag{40}$$

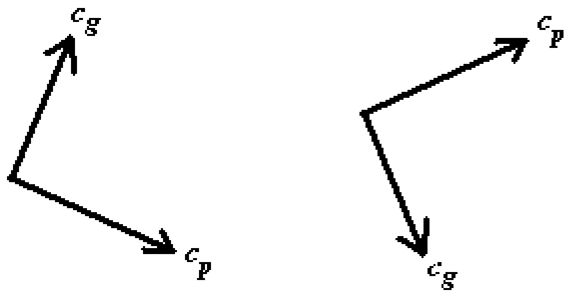
The group velocity is

$$\vec{c}_g = \frac{\partial\omega}{\partial k}\hat{i} + \frac{\partial\omega}{\partial l}\hat{j} + \frac{\partial\omega}{\partial m}\hat{k} = \pm \frac{m^2 N}{\mathrm{K}_H \mathrm{K}^3}\left(k\hat{i} + l\hat{j} - \frac{\mathrm{K}_H^2}{m}\hat{k}\right). \tag{41}$$

Inspection of (40) and (41) shows a curious fact that for internal waves, if there is a downward component to the phase velocity, then there is an upward component to the group velocity, and vice-versa. In fact, by taking the dot product of (40) and (41) we find that

$$\vec{c} \bullet \vec{c}_g = 0\,. \tag{42}$$

which shows that ***the group velocity and phase velocity are actually oriented at 90° to each other in the vertical plane!*** This is illustrated in the diagram below.



The link below contains an animated GIF loop showing internal wave dispersion for a wave number pointing toward the upper right. Note that individual crests propagate toward the upper-right corner, while the groups of waves propagate toward the lower-right corner. http://www.atmos.millersville.edu/~adecaria/ESCI343/internal-wave-loop.gif.

EXERCISES

1. Show that for an ideal gas

$$\frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz} \equiv -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz} - \frac{g^2}{c_s^2}\ .$$

2. Substitute sinusoidal solutions into equations (35) and (36) to derive the dispersion relation for internal gravity waves,

$$\omega^2 = \frac{K_H^2 N^2}{K^2}.$$

3. a. Show that the group velocity for internal waves is

$$\vec{c}_g = \pm\frac{m^2 N}{K_H K^3}\left(k\hat{i} + l\hat{j} - \frac{K_H^2}{m}\hat{k}\right).$$

b. What is the magnitude of the group velocity for purely vertically propagating waves?

c. What is the magnitude of the group velocity for purely horizontally propagating waves?

4. Use equations (40) and (41) to show that for internal waves, $\vec{c}\bullet\vec{c}_g = 0$.

5. For an ideal, incompressible gas the linearized governing equations in the *x*-*z* plane can be written as

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \quad \textbf{(a)}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + \frac{\theta'}{\bar{\theta}}g \quad \textbf{(b)}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad \textbf{(c)}$$

$$\theta' = -\frac{\partial \bar{\theta}}{\partial z}\Delta z \quad \textbf{(d)}$$

a. Substitute (d) into (b) and then take $\partial/\partial t$ of the resulting equation to get

$$\frac{\partial^2 w'}{\partial t^2} = -\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial t \partial z} - N^2 w' \quad \textbf{(e)}$$

b. Substitute sinusoidal solutions into (a), (c), and (e) to find the dispersion relation and phase speed.

c. What kind of waves are these?

6. Show that for the atmosphere, $\frac{g}{\rho}\frac{d\rho}{dz} >> \frac{g^2}{c_s^2}$.