ESCI 343 – Atmospheric Dynamics II
Lesson 8 – Sound Waves

References: *An Introduction to Dynamic Meteorology (3rd edition),* J.R. Holton
Waves in Fluids, J. Lighthill

SOUND WAVES

We will limit our analysis to sound waves traveling only along the *x*-axis, but keep in mind that we could easily extend this to waves traveling in an arbitrary direction. We start with the linearized equations of motion for the case of zero mean flow, and for which the reference density is constant with height. These are

$$\begin{aligned} \frac{\partial u'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\ \frac{\partial \rho'}{\partial t} &= -\bar{\rho}\frac{\partial u'}{\partial x} \end{aligned} \qquad \textbf{(1)}$$

We also have an equation of state that relates any three of the thermodynamic variables. We will use ρ, p, and θ as our thermodynamic variables, so our equation of state can be written as

$$p = p(\rho, \theta).$$

The equation of state can be written in differential form as

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_\theta d\rho + \left(\frac{\partial p}{\partial \theta}\right)_\rho d\theta. \qquad \textbf{(2)}$$

Sound waves are adiabatic, so that θ is constant. Therefore, we can write

$$\frac{dp}{dt} = \left(\frac{\partial p}{\partial \rho}\right)_\theta \frac{d\rho}{dt}$$

or in linearized form

$$\frac{\partial p'}{\partial t} = \left(\frac{\partial p}{\partial \rho}\right)_\theta \frac{\partial \rho'}{\partial t}. \qquad \textbf{(3)}$$

If we substitute this into the continuity equation, the linearized set of equations for one-dimensional sound waves are then

$$\begin{aligned} \frac{\partial u'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\ \frac{\partial p'}{\partial t} &= -\bar{\rho}\left(\frac{\partial p}{\partial \rho}\right)_\theta \frac{\partial u'}{\partial x} \end{aligned} \qquad \textbf{(4)}$$

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To find the dispersion relation for sound waves we assume sinusoidal solutions of

$$u' = Ae^{i(kx-\omega t)}$$
$$p' = Be^{i(kx-\omega t)} \quad \textbf{(5)}$$

and substitute them into the two prior equations to find that these are nondispersive waves travelling at a phase speed of

$$c = \pm\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_\theta} . \quad \textbf{(6)}$$

Since we know that these are sound waves, we have shown that for a general fluid the speed of sound is given by

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_\theta . \quad \textbf{(7)}$$

This shows us that the speed of sound is fundamental property of a fluid. It also shows us that sound waves are non-dispersive.[1]

THE CONTINUITY EQUATION WRITTEN WITH THE SPEED OF SOUND

Using Eqs. (2) and (7) we can write

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{Dp}{Dt} \quad \textbf{(8)}$$

which allows us to express the fully compressible continuity equation in terms of the material derivative of pressure,

$$\frac{1}{c_s^2}\frac{Dp}{Dt} = \rho\nabla \bullet \vec{V} . \quad \textbf{(9)}$$

This is a common way of writing the continuity equation.

THE SPEED OF SOUND IN AN IDEAL GAS

In an ideal gas, the equation of state has the form

$$p = \rho R'T . \quad \textbf{(10)}$$

In terms of potential temperature, this can be written as

$$p = \rho R'\theta\left(\frac{p}{p_0}\right)^{R'/c_p} . \quad \textbf{(11)}$$

[1] Actually, we've only shown that linear sound waves are nondispersive. We haven't, and won't discuss any of the effects of non-linear sound waves.

Taking the partial derivative of this with respect to density at constant potential temperature, and making use of the fact that for an ideal gas $c_p = c_v + R'$, after some patience you can find that for an ideal gas, the speed of sound is given by

$$c_s = \sqrt{\gamma R'T} \tag{12}$$

$$\gamma = \frac{c_p}{c_v}. \tag{13}$$

SOUND WAVES WITH A NON-ZERO MEAN FLOW

So far we've ignored the mean flow. If there is a basic state mean flow then the analysis is slightly more complex. Our linearized equations of motion become

$$\begin{aligned} &\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\ &\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x}\right) = -\bar{\rho}\frac{\partial u'}{\partial x} \end{aligned} \tag{14}$$

Remember that our goal is to find the dispersion relation for the waves. We do so by assuming a sinusoidal form for both dependent variables, of the form of Eqs. (5) and substitute these directly into Eqs. (14). The result is the following dispersion relation,

$$\begin{aligned} &\omega = k\left(\bar{u} \pm c_s\right) \\ &c = \bar{u} \pm c_s \end{aligned} \tag{15}$$

Note that the effect of the mean flow is additive. This is a property of linear waves. ***For linear waves, the phase speed with mean flow is just the phase speed without the mean flow plus the mean flow itself.***

VERTICALLY PROPOGATING SOUND WAVES

For sound waves propagating in all three dimensions the linearized governing equations are

$$\begin{aligned} &\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\ &\frac{\partial v'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} \\ &\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}}g \\ &\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t} + w'\frac{\partial \bar{p}}{\partial z}\right) = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right). \end{aligned} \tag{16}$$

For sufficiently large wave number (sufficiently small wavelengths) it turns out that we can ignore the effects of buoyancy (gravity) and the vertical gradient of the reference pressure. In practical terms this means as long as the waves are sufficiently short such that the wavelength is small compared to the scale over which pressure and density change with height, or

$$\mathrm{K} >> -\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial z} \qquad \textbf{(17)}$$

(details can be found in Lighthill, Section 4.2). Condition (17) can be expressed as

$$\lambda << 2\pi H_{\rho} \qquad \textbf{(18)}$$

where H_{ρ} is the ***density scale height*** of the atmosphere.[2] So, as long as we limit ourselves to sound waves in the normal range of human hearing we can ignore the effects of gravity on sound waves. Our equations are then

$$\begin{aligned}
\frac{\partial u'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \\
\frac{\partial v'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} \\
\frac{\partial w'}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} \\
\frac{1}{c_s^2}\frac{\partial p'}{\partial t} &= -\bar{\rho}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right).
\end{aligned} \qquad \textbf{(19)}$$

Substituting sinusoidal forms for u', v', w', and ρ' of

$$\begin{aligned}
u' &= Ae^{i(kx+ly+mz-\omega t)} \\
v' &= Be^{i(kx+ly+mz-\omega t)} \\
w' &= Ce^{i(kx+ly+mz-\omega t)} \\
p' &= De^{i(kx+ly+mz-\omega t)}
\end{aligned} \qquad \textbf{(20)}$$

yields the following dispersion relation

$$\omega^2 = c_s^2\left(k^2+l^2+m^2\right) = c_s^2 K^2 \, . \qquad \textbf{(21)}$$

[2] Density scale height is the *e*-folding scale for density, i.e., the altitude at which density is 37% of the surface value.

EXERCISES

1. The linearized governing equations for one-dimensional sound waves with zero mean flow are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

$$\frac{1}{c_s^2}\frac{\partial p'}{\partial t} = -\bar{\rho}\frac{\partial u'}{\partial x}$$

Substitute the assumed solutions

$$u' = Ae^{i(kx-\omega t)}$$

$$p' = Be^{i(kx-\omega t)}$$

into these equations to derive the dispersion relation for sound waves.

2. The linearized governing equations for three-dimensional sound waves with zero mean flow are

$$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}$$

$$\frac{1}{c_s^2}\frac{\partial p'}{\partial t} = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)$$

Derive the dispersion relation

$$\omega^2 = c_s^2\left(k^2 + l^2 + m^2\right) = c_s^2 \mathrm{K}^2$$

for these waves.

3. Show that the speed of sound in an ideal gas is

$$c_s = \pm\sqrt{\gamma R' T}\ .$$

4. Show that for an isothermal atmosphere that Condition (17) becomes Condition (18).

5. a. Find the scale height and speed of sound for an isothermal atmosphere with a temperature of 255K.

b. For this atmosphere, find out how large the wavelength of an acoustic wave would need to be before we started concerning ourselves with the effects of gravity and buoyancy on these waves.