ESCI 343 – Atmospheric Dynamics II
Lesson 7 –Gravity Waves in a Two-layer Fluid

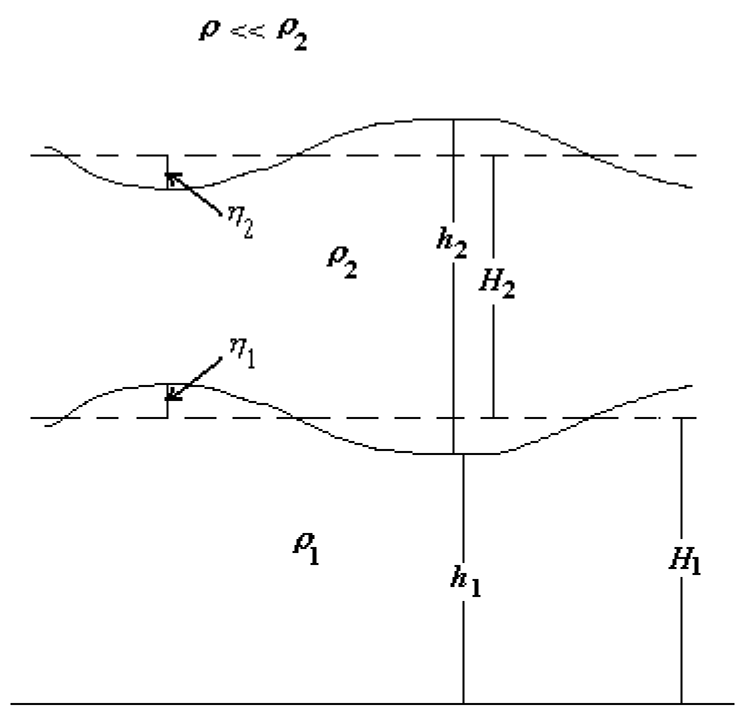
References: *Atmosphere-Ocean Dynamics*, A.E. Gill

GENERAL

In a two-layer fluid such as that shown in the diagram below, gravity waves can exist on either the interface at the top of the upper fluid, or on the interface between the two fluids. We assume that the space above the uppermost fluid is filled with a fluid of very low density and very large depth, so that at the upper interface we can use *g* instead of *g*′, and that p_0 at the upper interface is zero (we will refer to this upper interface as the *free surface*). The gravity waves that form on the free surface and the interface are not independent of one another.

DERIVATION OF DISPERSION RELATION FOR TWO-LAYER FLUID

We can examine their dependence by applying the shallow-water equations to each layer of fluid.



Equations in the upper fluid. In the upper fluid the pressure gradient force is solely due to the slope of the free surface, and the shallow-water momentum equations (ignoring rotation) are

$$\begin{aligned}\frac{Du_2}{Dt} &= -g\frac{\partial \eta_2}{\partial x}\\ \frac{Dv_2}{Dt} &= -g\frac{\partial \eta_2}{\partial y}\,.\end{aligned} \tag{1}$$

The continuity equation in the upper fluid is

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0 \tag{2}$$

which when integrated over the depth of the upper fluid (h_2) becomes

$$\frac{D\eta_2}{Dt} - \frac{D\eta_1}{Dt} = -h_2\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right). \tag{3}$$

Equations in the lower fluid. In the lower fluid the pressure gradient is not only due to the slope of the interface, but also due to the slope of the free surface. This is seen by finding the pressure at a point at height z in the lower fluid, which is

$$p = p_0 + \rho_2 g\eta_2 + \rho_2 gH_2 + \rho_1 g(H_1 - z) + \rho_1 g\eta_1 - \rho_2 g\eta_1 \tag{4}$$

so that

$$\frac{1}{\rho_1}\frac{\partial p}{\partial x} = \frac{\rho_2}{\rho_1} g\frac{\partial \eta_2}{\partial x} + g'\frac{\partial \eta_1}{\partial x}. \tag{5}$$

Therefore, the momentum equations in the lower fluid are

$$\begin{aligned}\frac{Du_1}{Dt} &= -\frac{\rho_2}{\rho_1} g\frac{\partial \eta_2}{\partial x} - g'\frac{\partial \eta_1}{\partial x}\\ \frac{Dv_1}{Dt} &= -\frac{\rho_2}{\rho_1} g\frac{\partial \eta_2}{\partial y} - g'\frac{\partial \eta_1}{\partial y}.\end{aligned} \tag{6}$$

The continuity equation in the lower layer is

$$\frac{D\eta_1}{Dt} = -h_1\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right). \tag{7}$$

DISPERSION RELATION FOR SHALLOW-WATER GRAVITY WAVES IN A TWO-LAYER FLUID

The linearized shallow-water equations for a two-layer fluid with zero mean flow, and for waves traveling in the x-direction only, are

$$\frac{\partial u_2'}{\partial t} = -g\frac{\partial \eta_2}{\partial x} \tag{8}$$

$$\frac{\partial \eta_2}{\partial t} - \frac{\partial \eta_1}{\partial t} = -H_2\frac{\partial u_2'}{\partial x} \tag{9}$$

$$\frac{\partial u_1'}{\partial t} = -\frac{\rho_2}{\rho_1} g\frac{\partial \eta_2}{\partial x} - g'\frac{\partial \eta_1}{\partial x} \tag{10}$$

$$\frac{\partial \eta_1}{\partial t} = -H_1\frac{\partial u_1'}{\partial x} \tag{11}$$

To find the dispersion relation we could proceed by assuming sinusoidal functions for the four dependent variables and substituting them into the governing equations to get a set of four algebraic equations. This would be very messy, since we would have to find the determinant of a 4×4 matrix. Instead, we will try to reduce the four equations and four unknowns into a system of two equations and two unknowns.

Eliminating the velocity component from equations (8) and (9) gives

$$\frac{\partial^2 \eta_1}{\partial t^2} = \frac{\partial^2 \eta_2}{\partial t^2} - gH_2 \frac{\partial^2 \eta_2}{\partial x^2}. \tag{12}$$

Eliminating the velocity component from equations (10) and (11) gives

$$\frac{\partial^2 \eta_1}{\partial t^2} - g'H_1 \frac{\partial^2 \eta_1}{\partial x^2} = \frac{\rho_2}{\rho_1} gH_1 \frac{\partial^2 \eta_2}{\partial x^2}. \tag{13}$$

Assuming sinusoidal forms for η_1 and η_2 of

$$\begin{aligned} \eta_1 &= Ae^{i(kx-\omega t)} \\ \eta_2 &= Be^{i(kx-\omega t)} \end{aligned} \tag{14}$$

into equations (12) and (13) yields the following algebraic set of equations,

$$\begin{aligned} &\omega^2 A - \left(\omega^2 - gH_2k^2\right)B = 0 \\ &\left(\omega^2 - g'H_1k^2\right)A - \left(\rho_2/\rho_1\right)gH_1k^2 B = 0 \end{aligned} \tag{15}$$

which gives a fourth-order polynomial for ω,

$$\omega^4 - gHk^2\omega^2 + g'gH_1H_2k^4 = 0 \tag{16}$$

where H is the total depth defined as

$$H \equiv H_1 + H_2. \tag{17}$$

In terms of phase speed, c, the fourth-order polynomial is

$$c^4 - gHc^2 + g'gH_1H_2 = 0 \tag{18}$$

which solved for c^2 is

$$c^2 = \frac{gH}{2} \pm \frac{1}{2}\sqrt{g^2H^2 - 4g'gH_1H_2} \tag{19}$$

or

$$c^2 = gH\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4g'H_1H_2}{gH^2}}\right). \tag{20}$$

Equation (20) has two roots for c^2. This indicates that the two-layer fluid supports two types (or modes) of wave motion, one associated with each root. The ratio of the disturbance amplitude on the free surface to that of the interior interface is found from equation (5.a) to be

$$\frac{B}{A} = \frac{\omega^2}{\omega^2 - gH_2k^2} = \frac{c^2}{c^2 - gH_2} \tag{21}$$

STRUCTURE OF THE TWO MODES

The speed of shallow-water gravity waves in a two-layer fluid is

$$c^2 = gH\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4g'H_1H_2}{gH^2}}\right). \tag{20}$$

Equation (20) has two roots for c. This indicates that the two-layer fluid supports two types (or modes) of wave motion, one associated with each root. The ratio of the disturbance amplitude on the free surface (B) to that of the interior interface (A) is

$$\frac{B}{A} = \frac{\omega^2}{\omega^2 - gH_2k^2} = \frac{c^2}{c^2 - gH_2} \tag{22}$$

The two-modes supported by the two-layer fluid are very different. We can best study them by assuming that the densities of the two fluids differ only slightly. This means that

$$\frac{g'}{g} = \frac{\rho_1 - \rho_2}{\rho_1} << 1$$

so that

$$\frac{4g'H_1H_2}{gH^2} << 1.$$

Through Taylor series expansion it can be shown for $x << 1$ that

$$\sqrt{1-x} \cong 1 - \frac{1}{2}x.$$

This allows us to write the phase speed of the wave (from 6) as

$$c^2 \cong gH\left[\frac{1}{2} \pm \frac{1}{2}\left(1 - \frac{2g'H_1H_2}{gH^2}\right)\right] \tag{23}$$

Structure of mode with positive root. For the mode with the positive root, the phase speed is

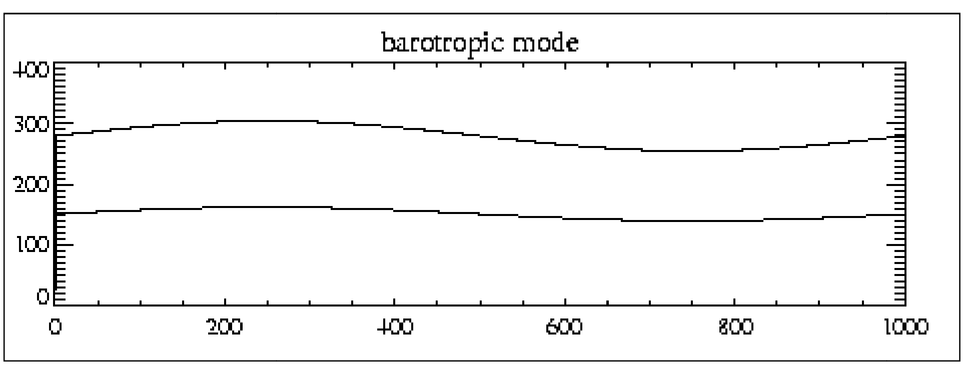
$$c^2 = gH\ , \tag{24}$$

which is identical to that of an external gravity wave on the surface of a homogeneous fluid of depth H. The disturbances on the two interfaces are seen to be in phase, as is shown from substituting (24) into (22) to get

$$\frac{B}{A} = \frac{H}{H_1}.$$

This also shows that the disturbance on the free surface is larger in amplitude than the disturbance on the interior interface (since the total depth of the fluid, H, is greater than the depth of the lower layer, H_1).

For this mode, the u-components of velocity are also in phase and are nearly equal, leading to the general name for this mode as the *barotropic mode*. The structure of this mode is shown below.



Structure of mode with negative root. For the mode with the negative root, the phase speed is

$$c^2 = g' \frac{H_1 H_2}{H}, \tag{25}$$

The ratio of the disturbance amplitudes on the two interfaces are found by substituting (25) into (22) to get

$$\frac{B}{A} = \frac{g' H_1 H_2}{g' H_1 H_2 - gHH_2},$$

which can be written as

$$\frac{B}{A} = \frac{\frac{g' H_1 H_2}{gHH_2}}{\frac{g' H_1 H_2}{gHH_2} - 1} = \frac{\frac{g' H_1}{gH}}{\frac{g' H_1}{gH} - 1};$$

since

$$\frac{g' H_1}{gH} << 1,$$

and for small *x*,

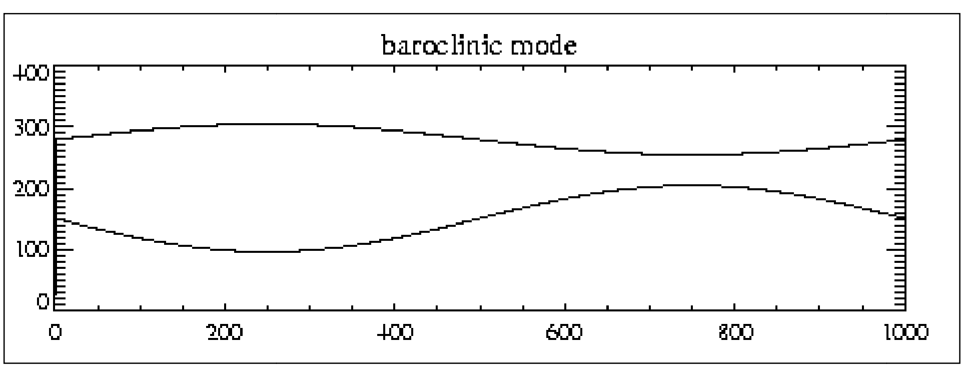
$$\frac{x}{x-1} \cong -x,$$

we have that for the negative root,

$$\frac{B}{A} \cong -\frac{g'}{g} \frac{H_1}{H}. \tag{26}$$

This also shows that the disturbance on the free surface is much smaller in amplitude than the disturbance on the interior interface, and is of opposite sign, so the two interfaces are 180° out of phase.

For this mode, the *u*-components of velocity are also 180° out of phase, leading to the general name for this mode as the *baroclinic mode*. The structure of this mode is shown below.



MULTIPLE LAYERED FLUIDS

We've that for a two-layer, shallow-water fluid there are two distinct modes, a barotropic and a baroclinic mode. In a three layer fluid it turns out there are three modes: a barotropic mode and two baroclinic modes. In general, an *n*-layer fluid will have *n* modes: a barotropic mode, and *n*−1 baroclinic modes. A fluid with continuous stratification can therefore be expected to have an infinite number of baroclinic modes.

EQUIVALENT DEPTH

One concept from the discussion of two-layer fluids will be important to us later when discussing internal waves in either a multi-layered, or continuously stratified fluid. That is the concept of *equivalent depth*. From the phase speed for the baroclinic mode in a two-layer fluid,

$$c^2 = g'\frac{H_1 H_2}{H},$$

we see that the speed of this mode is equal to that of an external gravity wave in a uniform fluid having a depth equal to

$$c^2 = gH_e \tag{27}$$

where

$$H_e \equiv \frac{g'}{g}\frac{H_1 H_2}{H}. \tag{28}$$

H_e is called the *equivalent depth*, and is the depth that the fluid would have to have in order for an external gravity wave to have the same speed as the baroclinic mode. The equivalent depth is important, because if we can calculate the equivalent depth for the fluid we know that the phase speed of the baroclinic mode is given by (27).

For a fluid with multiple layers, there are multiple baroclinic modes. However, each mode will have an equivalent depth, and once we know that, we know the dispersion relation! In a fluid with continuous stratification there would be an infinite number of baroclinic modes; however, not all of the modes will be important (some will be very weak), and if we can identify the most important baroclinic modes (maybe as few as two or three) and can find the equivalent depths for these modes, then we know the dispersion relations for these modes! This type of analysis is often done in both the atmosphere and the ocean, and is known as *normal mode* analysis.

WHY WE STUDY SHALLOW-WATER GRAVITY WAVE THEORY

As meteorologists, why do we bother studying shallow-water gravity waves? After all, the atmosphere is neither shallow, nor water! The answer is, ***"because of the concept of equivalent depth."***

The atmosphere is a multi-layer fluid, and supports many baroclinic modes of oscillation. For the scales of motion studied in synoptic-scale dynamics the hydrostatic assumption is valid. Recall that shallow-water wave theory also assumes hydrostatic balance. If we can find the equivalent depths for each of the baroclinic modes of oscillation, then we can use the shallow-water gravity wave dispersion relation,

$$\begin{aligned} \omega^2 &= gH_e k^2 \\ c^2 &= gH_e \end{aligned} \tag{29}$$

to study the linear gravity waves that are supported by the atmosphere.