ESCI 343 – Atmospheric Dynamics II
Lesson 6 – Shallow-water Surface Gravity Waves

References: *An Introduction to Dynamic Meteorology* (3rd edition), J.R. Holton
Numerical Prediction and Dynamic Meteorology (2nd edition), G.J. Haltiner and R.T. Williams
Waves in Fluids, J. Lighthill

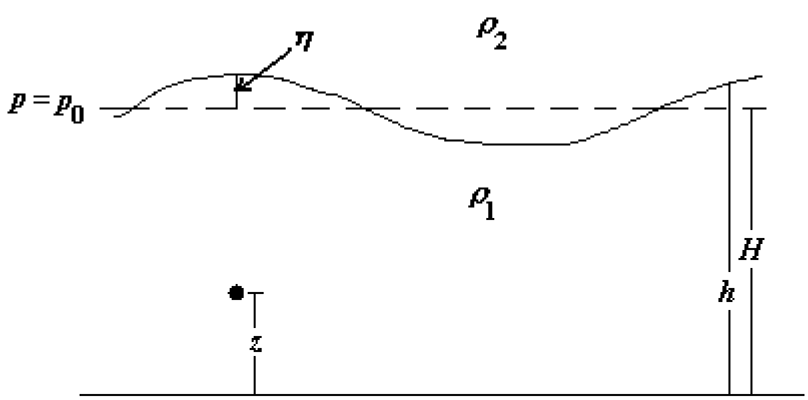
GENERAL

Surface gravity waves are waves on the surface of a liquid, the restoring force of which is gravity. These waves are familiar to all of us as the waves on the ocean or a lake. Though pure surface gravity waves do not appear in the atmosphere, their study is a useful prelude to the study of other gravity wave types in the atmosphere.

We will first limit our study to surface gravity waves on the free surface of a constant density fluid. We will also limit our study to waves in hydrostatic balance. This assumption implies that we are studying waves whose wavelength is much larger than the depth of the fluid (remember the condition from scale analysis for assuming hydrostatic balance is that the horizontal length scale be much larger than the vertical length scale). Thus we are limited to either very-long wavelengths, or very shallow water. This is alternately known as either the *shallow-water* approximation or the *long-wave* approximation.

THE SHALLOW-WATER MOMENTUM EQUATIONS

The diagram below shows the interface between two fluids of different, constant densities. The dashed line shows the position of the interface if the fluids are undisturbed. The solid line shows the interface displaced. The depth of the lower fluid is H.



If we assume that the upper fluid is very deep compared to the displacement of the interface, η (note that $H + \eta = h$), then we can assume that the pressure at the level of the undisturbed interface (the dashed line in the figure) remains constant at a value of p_0[1]. If

[1] We have to assume that the upper fluid is deep in order to assume that p_0 is constant. This is because a displacement of the interface upward results in either divergence or convergence in the upper fluid as it adjusts to the change in interface height. This means that there would be horizontal flow in the upper fluid, which would require a horizontal pressure gradient in the upper fluid. By constraining our

the lower fluid is in hydrostatic balance, then the pressure at any point in the lower fluid is proportional to the weight of the fluid above it. Therefore, at the point shown in the diagram the pressure will be

$$p = p_0 + \rho_1 g (H - z) + \rho_1 g \eta - \rho_2 g \eta \tag{1}$$

and the horizontal pressure gradient force will be

$$-\frac{1}{\rho_1}\frac{\partial p}{\partial x} = -\left(\frac{\rho_1 - \rho_2}{\rho_1}\right) g \frac{\partial \eta}{\partial x} . \tag{2}$$

Since

$$h = H + \eta \tag{3}$$

then

$$\frac{\partial h}{\partial x} = \frac{\partial \eta}{\partial x} \tag{4}$$

so

$$-\frac{1}{\rho_1}\frac{\partial p}{\partial x} = -\left(\frac{\rho_1 - \rho_2}{\rho_1}\right) g \frac{\partial h}{\partial x} . \tag{5}$$ [2]

The quantity

$$g' = \left(\frac{\rho_1 - \rho_2}{\rho_1}\right) g \tag{6}$$

is called *reduced gravity*, and the momentum equations for the lower fluid are written as

$$\begin{aligned} \frac{Du}{Dt} &= -g' \frac{\partial h}{\partial x} + fv \\ \frac{Dv}{Dt} &= -g' \frac{\partial h}{\partial y} - fu . \end{aligned} \tag{7}$$

If the two fluids are greatly different in densities (such as air and water), then $\rho_1 - \rho_2 \cong \rho_1$, and $g' \cong g$ (note that the prime on g does not refer to a perturbation).

Since we've assumed that the lower fluid is in hydrostatic balance, we've constrained our analysis to motions whose horizontal scale is much greater than the vertical scale (the depth of the fluid). For this reason, we refer to equation set (7) as the *shallow-water* momentum equations.

Note that the pressure gradient force at any point in the lower fluid is independent of depth! This means that the fluid motion is also independent of depth. Therefore, the lower fluid is *barotropic*.

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discussion to a very deep upper fluid, there is minimal convergence or divergence in the upper fluid (since the amount of mass replaced is minimal compared to the overall mass in the fluid column), and therefore, minimal horizontal flow in the upper fluid.

[2] For convenience we derived this expression as though *H* were constant in *x* and *y*. However, if *H* is allowed to vary in *x* and *y* we would obtain the same result. The derivation would just be a little more cumbersome.

THE SHALLOW-WATER CONTINUITY EQUATION

The continuity equation in the lower fluid is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,. \tag{8}$$

If we integrate the continuity equation from the bottom of the fluid to the interface we get

$$\int_0^h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz+\int_0^h\frac{\partial w}{\partial z}\,dz=0 \tag{9}$$

which becomes

$$w(h)-w(0)=-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{10}$$

The vertical velocity at the bottom of the fluid is zero. Also,

$$w(h)=\left.\frac{Dz}{Dt}\right|_h=\frac{Dh}{Dt} \tag{11}$$

so the shallow-water continuity equation is

$$\frac{Dh}{Dt}=-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{12}$$

Note: Equation (12) was derived for a flat bottom, but it actually applies to a fluid with a non-flat bottom. In this case, h is the total depth of the fluid from bottom to top.

LINEARIZED SHALLOW-WATER EQUATIONS

To linearize the shallow-water equations we use the following perturbation forms for the dependent variables

$$\begin{aligned} u&=\bar{u}+u'\\ v&=\bar{v}+v'\\ h&=H(x,y)+\eta-z_0. \end{aligned} \tag{13}$$

Note that H is the mean undisturbed height of the fluid surface, which can depend on x, and y. Thus, the undisturbed surface of the fluid can be sloped to support a geostrophically-balanced mean flow. The term z_0 is the elevation of the bottom topography, and for a flat bottom would be zero.

We will assume a flat bottom ($z_0=0$), and also assume that the base state is in geostrophic balance so that

$$\begin{aligned} \bar{u}&=-\frac{g}{f}\frac{\partial H}{\partial y}\\ \bar{v}&=\frac{g}{f}\frac{\partial H}{\partial x}, \end{aligned} \tag{14}$$

Putting (13) into equations (7) and (12), ignoring products of perturbations, and using (14), we get

$$
\begin{aligned}
&\frac{\partial u'}{\partial t}+\bar{u}\,\frac{\partial u'}{\partial x}+\bar{v}\,\frac{\partial u'}{dy}=-g'\frac{\partial \eta}{\partial x}+fv' \\
&\frac{\partial v'}{\partial t}+\bar{u}\,\frac{\partial v'}{\partial x}+\bar{v}\,\frac{\partial v'}{dy}=-g'\frac{\partial \eta}{\partial y}-fu' \\
&\frac{\partial \eta}{\partial t}+\bar{u}\left(\frac{\partial H}{\partial x}+\frac{\partial \eta}{\partial x}\right)+\bar{v}\left(\frac{\partial H}{\partial y}+\frac{\partial \eta}{dy}\right)+u'\frac{\partial H}{\partial x}+v'\frac{\partial H}{\partial y}=-H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right).
\end{aligned}
\tag{15}
$$

Equations (15) are the linearized shallow-water equations.

Note: Usually we work with cases where $\frac{\partial H}{\partial x} << \frac{\partial \eta}{\partial x}$ and $\frac{\partial H}{\partial y} << \frac{\partial \eta}{\partial y}$. This allows the $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$ terms to be omitted in the linearized continuity equation, which then simply becomes

$$
\frac{\partial \eta}{\partial t}+\bar{u}\,\frac{\partial \eta}{\partial x}+\bar{v}\,\frac{\partial \eta}{dy}=-H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right).
\tag{16}
$$

DISPERSION RELATION FOR SHALLOW-WATER GRAVITY WAVES

To find the dispersion relation for shallow-water gravity waves in the absence of a mean flow, with constant *H*, and ignoring Coriolis, we start with

$$
\begin{aligned}
&\frac{\partial u'}{\partial t}=-g'\frac{\partial \eta}{\partial x} \\
&\frac{\partial v'}{\partial t}=-g'\frac{\partial \eta}{\partial y} \\
&\frac{\partial \eta}{\partial t}=-H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right).
\end{aligned}
\tag{17}
$$

Assuming a sinusoidal disturbance such as

$$
\begin{aligned}
u' &= Ae^{i(kx+ly-\omega t)} \\
v' &= Be^{i(kx+ly-\omega t)} \\
\eta &= Ce^{i(kx+ly-\omega t)}
\end{aligned}
\tag{18}
$$

and substituting into (17), we get the following matrix equation

$$
\begin{pmatrix} \omega & 0 & -g'k \\ 0 & \omega & -gl \\ kH & lH & -\omega \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\tag{19}
$$

In order that *A*, *B*, and *C* not be zero, then

$$\begin{vmatrix} \omega & 0 & -g'k \\ 0 & \omega & -g'l \\ kH & lH & -\omega \end{vmatrix} = 0\,, \tag{20}$$

and solving this for ω gives

$$\omega = \sqrt{\left(k^2 + l^2\right) g'H}\ . \tag{21}$$

The total wave number in the direction of propagation is given by

$$K^2 = k^2 + l^2 \tag{22}$$

so we get the following dispersion relation and phase speed,

$$\begin{aligned} \omega &= K\sqrt{g'H} \\ c &= \frac{\omega}{K} = \sqrt{g'H}\ . \end{aligned} \tag{23}$$

These waves are nondispersive since the wavenumber does not appear on the right-hand side of (23).

EXERCISES

1. Find the dispersion relation for one-dimensional (*x*-direction) shallow-water gravity waves with a non-zero mean flow in the zonal direction (i.e., $\bar{u} \neq 0$, $\bar{v} = 0$).

2. a. Find the dispersion relation for two-dimensional (*x* and *y*-directions) shallow-water gravity waves with a zero mean flow, but including the Coriolis parameter (these are known as *shallow-water, inertial-gravity waves*).

b. Find the group velocity and phase speed of these waves. Are they dispersive?

3. The general dispersion relation for one-dimensional surface gravity waves (not restricted to shallow water) traveling in the *x*-direction is

$$\omega = \bar{u}k \pm \sqrt{gk \tanh kH} \,.$$

a. What is the phase speed for these waves?

b. Are these waves dispersive?

4. a. For very short waves, or for very deep water, ($kH >> 1$). Show that in this case the dispersion relation for surface gravity waves is

$$\omega = \bar{u}k \pm \sqrt{gk} \,.$$

(This is known as the *short-wave approximation*, or *deep-water approximation*). Note that for $x >> 1$, $\tanh x \cong 1$.

b. What is the group velocity for these waves?

c. Are these waves dispersive?

5. a. For very long waves, or for very shallow water, ($kH << 1$). Show that in this case the dispersion relation for surface gravity waves is

$$\omega = \bar{u}k \pm k\sqrt{gH} \,.$$

(This is known as the *long-wave approximation*, or *shallow-water approximation*). Note that for $x << 1$, $\tanh x \cong x$.

b. What is the group velocity for these waves?

c. Are these waves dispersive?

6. Calculate the speed of a shallow-water surface gravity wave for a fluid having a depth equal to the scale height of the atmosphere (~8.1 km) (assume zero mean flow). How does this compare with the speed of sound?