

ESCI 343 – Atmospheric Dynamics II

Lesson 5 – Linear Waves

Reference: *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
Waves in Fluids, J. Lighthill
Atmosphere-Ocean Dynamics, A.E. Gill

Reading: Holton, Section 7.2

LINEAR VERSUS NONLINEAR WAVES

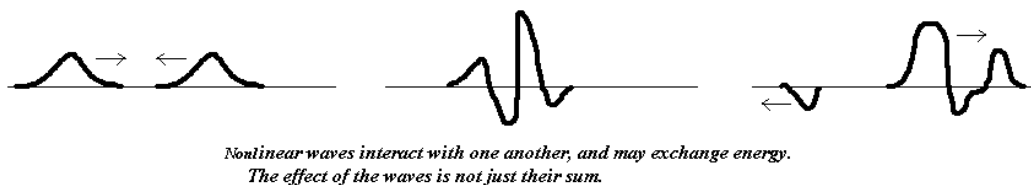
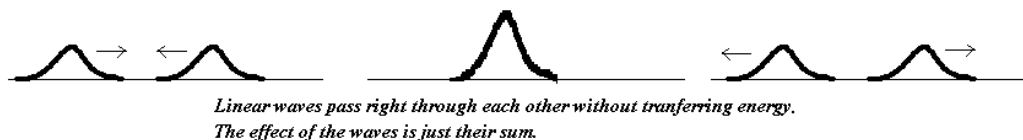
A linear set of equations does not contain products of the dependent variables. A non-linear set of equation contains products of dependent variables. The momentum equations are definitely nonlinear, since the advective terms look like

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

which has products of the dependent variables u , v , and w .

A linear set of equations support linear waves, while a nonlinear set of equations supports nonlinear waves. ***Linear waves and nonlinear waves behave very differently!*** The primary difference is that ***linear waves do not interact with one another, and can't exchange energy!*** Two linear waves will pass right through one another. Any interference between the two waves is strictly linear, meaning at a given point, the effect of the wave is just the sum of the effects of the two waves. ***Nonlinear waves interact and may exchange energy!*** The diagram below illustrates this.

Nonlinear waves are much more complex, and more difficult to study, than are linear waves. Unfortunately, the governing equations are highly non-linear (due to the advective terms), and therefore, atmospheric waves are nonlinear.



THE PERTURBATION METHOD

The governing equations are nonlinear. In order to study the properties of atmospheric waves we “linearize” the governing equations, and then study the linear waves supported by these equations. By studying these linear waves we hope to learn some information about the waves and their relevance.

In order to linearize the equations we use the *perturbation method*. We start by dividing all the dependent variables into two parts. The first part is known as the *basic state*, and is assumed to be either constant, or only a function of the spatial coordinates.

The second part is the *perturbation*, and is allowed to vary with time, and in all three space directions. For example,

$$\begin{aligned}u(x, y, z, t) &= \bar{u} + u'(x, y, z, t) \\v(x, y, z, t) &= \bar{v} + v'(x, y, z, t) \\w(x, y, z, t) &= w'(x, y, z, t); \quad \text{assume } \bar{w} = 0 \\p(x, y, z, t) &= \bar{p}(x, y, z) + p'(x, y, z, t) \\\rho(x, y, z, t) &= \bar{\rho}(x, y, z) + \rho'(x, y, z, t)\end{aligned}$$

Another assumption is that the basic state must satisfy the equations of motion when the perturbations are zero.

A third critical assumption for the perturbation method is that the perturbations must be small so that products of perturbations can be neglected. (Do not confuse this procedure with Reynolds averaging. Although the two procedures may look similar, they are really very different.)

We then take the divided dependent variables, substitute them into the equations, and multiply everything out. Since we can ignore terms that are the products of two perturbations, any such term can be crossed out.

THE PERTURBATION METHOD APPLIED TO THE *U*-MOMENTUM EQUATION

Applying the perturbation method to the *u*-momentum equation is illustrated below.

$$\frac{\partial(\bar{u} + u')}{\partial t} + (\bar{u} + u') \frac{\partial(\bar{u} + u')}{\partial x} + (\bar{v} + v') \frac{\partial(\bar{u} + u')}{\partial y} + (\bar{w} + w') \frac{\partial(\bar{u} + u')}{\partial z} = -\frac{1}{(\bar{\rho} + \rho')} \frac{\partial(\bar{p} + p')}{\partial x} + f(\bar{v} + v')$$

We can simplify this equation by recognizing that the basic state variables are independent of time, and that only the pressure and density basic state variables are functions of *z*. Also, we assumed that the base-state vertical velocity is zero. The equation then becomes

$$\frac{\partial u'}{\partial t} + (\bar{u} + u') \frac{\partial u'}{\partial x} + (\bar{v} + v') \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z} = -\frac{1}{(\bar{\rho} + \rho')} \frac{\partial(\bar{p} + p')}{\partial x} + f(\bar{v} + v')$$

Since the perturbation quantities are very small, we assume that we can ignore products of perturbation quantities. This further simplifies the equation to

$$\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + \bar{v} \frac{\partial u'}{\partial y} = -\frac{1}{(\bar{\rho} + \rho')} \frac{\partial(\bar{p} + p')}{\partial x} + f(\bar{v} + v')$$

We also assume that we can ignore perturbations of density in the horizontal pressure gradient term (similar to the Boussinesq approximation), to get

$$\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + \bar{v} \frac{\partial u'}{\partial y} = -\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x} - \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} + f\bar{v} + fv'$$

And finally, if we assume that the basic state is in geostrophic balance, then

$$-\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x} + f\bar{v} = 0,$$

so that we are left with

$$\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + \bar{v} \frac{\partial u'}{\partial y} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} + fv'.$$

This is the linearized, or perturbation form of, the u -momentum equation. Linearization of the v -momentum equation proceeds in a similar manner.

LINEARIZING THE w -MOMENTUM EQUATION

The w -momentum equation is a bit trickier, because we can't ignore the density perturbation in the vertical pressure gradient term like we could in the horizontal pressure gradient term of the u -momentum equation. So, after substituting the basic state and perturbation variables into the w -momentum equation we get

$$\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} + \bar{v} \frac{\partial w'}{\partial y} = -\frac{1}{(\bar{\rho} + \rho')} \frac{\partial (\bar{p} + p')}{\partial z} - g.$$

A rule of algebra tells us that if $a \ll 1$, then

$$\frac{1}{1+a} \cong 1-a$$

Using this rule we can write

$$\frac{1}{\bar{\rho} + \rho'} = \frac{1}{\bar{\rho}(1 + \rho'/\bar{\rho})} \cong \frac{1}{\bar{\rho}} \left(1 - \frac{\rho'}{\bar{\rho}}\right).$$

Using this, the RHS of the w -momentum equation becomes

$$-\frac{1}{\bar{\rho} + \rho'} \frac{\partial}{\partial z} (\bar{p} + p') - g = \frac{1}{\bar{\rho}} \left(\frac{\rho'}{\bar{\rho}} - 1 \right) \frac{\partial}{\partial z} (\bar{p} + p') - g = \frac{1}{\bar{\rho}} \left(\frac{\rho'}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} + \frac{\rho'}{\bar{\rho}} \frac{\partial p'}{\partial z} - \frac{\partial \bar{p}}{\partial z} - \frac{\partial p'}{\partial z} \right) - g$$

and since we can ignore products of perturbation terms, this simplifies to

$$\frac{1}{\bar{\rho}} \left(\frac{\rho'}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} - \frac{\partial \bar{p}}{\partial z} - \frac{\partial p'}{\partial z} \right) - g.$$

If the basic state is in hydrostatic balance, then

$$\frac{\partial \bar{\rho}}{\partial z} = -\bar{\rho} g .$$

Substituting this into the equation above it gives

$$\frac{1}{\bar{\rho}} \left(\frac{\rho'}{\bar{\rho}} (-\bar{\rho} g) - (-\bar{\rho} g) - \frac{\partial p'}{\partial z} \right) - g = -\frac{\rho'}{\bar{\rho}} g - \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z}$$

so that the linearized w -momentum equation is

$$\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} + \bar{v} \frac{\partial w'}{\partial y} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}} g .$$

Note that what we've done is to use the basic state density everywhere except in the buoyancy term (the term involving g), where we used the perturbation density. This is essentially the Boussinesq approximation, the difference being that the reference density is allowed to vary spatially, whereas in the Boussinesq approximation the reference density is assumed to be a true constant.

THE FINAL FORM OF THE PERTURBATION EQUATIONS

If we assume that the basic state is in geostrophic and hydrostatic balance, and that the base-state density is a function of z only, the linearized momentum and continuity equations are

$$\begin{aligned} \frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + \bar{v} \frac{\partial u'}{\partial y} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} + f v' \\ \frac{\partial v'}{\partial t} + \bar{u} \frac{\partial v'}{\partial x} + \bar{v} \frac{\partial v'}{\partial y} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial y} - f u' \\ \frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} + \bar{v} \frac{\partial w'}{\partial y} &= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}} g \\ \frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} + \bar{v} \frac{\partial \rho'}{\partial y} + w' \frac{d\bar{\rho}}{dz} &= -\bar{\rho} \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) \end{aligned}$$

THE GENERAL METHOD FOR FINDING THE DISPERSION RELATION

There is a general method for finding the dispersion relation for the waves supported by a linearized set of equations. The method is best illustrated by example. We will use the linearized, one-dimensional shallow-water equations given by

$$\begin{aligned}\frac{\partial u'}{\partial t} &= -g \frac{\partial h'}{\partial x} \\ \frac{\partial h'}{\partial t} &= -H \frac{\partial u'}{\partial x}\end{aligned}$$

as our example (here, g is the acceleration due to gravity, and the depth of the fluid, h is given by

$$h = H + h'.$$

These equations support shallow-water gravity waves.

1. Step 1 is to assume that all dependent variables have a sinusoidal form

$$u' = A e^{i(kx - \omega t)}$$

$$h' = B e^{i(kx - \omega t)}$$

2. Step 2 is to plug the assumed form of the dependent variables into the linearized governing equations. In our case this yields two algebraic equations in A and B .

$$\omega A - kgB = 0$$

$$kHA - \omega B = 0$$

3. Step 3 is to write these equations in matrix form

$$\begin{pmatrix} \omega & -kg \\ kH & -\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In order for these two equations to be linearly independent (i.e., not have a trivial solution of $A = B = 0$), the determinant of the coefficient matrix must equal zero

$$\begin{vmatrix} \omega & -kg \\ kH & -\omega \end{vmatrix} = 0$$

4. Step 4 is to take the determinant of the coefficient matrix and solve for ω . In our case this becomes

$$-\omega^2 + k^2 gH = 0$$

$$\omega = \pm k \sqrt{gH}$$

with a phase speed of

$$c = \frac{\omega}{k} = \pm \sqrt{gH}$$

and a group velocity of

$$c_g = \frac{\partial \omega}{\partial k} = \pm \sqrt{gH}$$

(Note that since the phase speed does not depend on k then these waves are nondispersive, also evident because the phase speed and group velocity are identical). This is the standard method of determining the dispersion relation for a set of equations, and will be applied to more complex equations.

EXERCISES

1. Assume that all base-state variables are constant except for density, which is a function of height only. Also assume the base-state vertical velocity is zero. Show that the linearized continuity equation is

$$\frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} + \bar{v} \frac{\partial \rho'}{\partial y} + w' \frac{d\bar{\rho}}{dz} = -\bar{\rho} \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right).$$

2. Show that if a fluid is incompressible then the linearized continuity equation is simply

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

3. a. Find the dispersion relation for waves supported by the shallow-equations with a mean flow,

$$\begin{aligned} \frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} &= -g \frac{\partial h'}{\partial x} \\ \frac{\partial h'}{\partial t} + \bar{u} \frac{\partial h'}{\partial x} &= -H \frac{\partial u'}{\partial x} \end{aligned}$$

and show that it is

$$\omega = k\bar{u} \pm k\sqrt{gH}$$

- b. What are the phase speed and group velocity of these waves?
- c. Are these waves dispersive?
- d. How does the phase speed and group velocity of these waves compare to shallow-water gravity waves without a mean flow?