ESCI 343 – Atmospheric Dynamics II
Answers to Selected Exercises for Lesson 9

1. Show that for an ideal gas

$$\frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz} \equiv -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz} - \frac{g^2}{c_s^2}.$$

Answer: Rewrite as

$$\frac{g}{\bar{\theta}}\frac{d\theta}{dz} = g\frac{d\ln\theta}{dz}.$$

Start with the ideal gas law and the definition of potential temperature

$$p = \rho R' T$$

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa}$$

and eliminate temperature to get

$$p = \rho R' \theta\left(\frac{p}{p_0}\right)^{\kappa}.$$

Take natural log to get

$$\ln p = \ln\rho + \ln R' + \ln\theta + \kappa\ln p - \kappa\ln p_0$$

or

$$\ln\theta = -\ln\rho + (1-\kappa)\ln p - \ln R' + \kappa\ln p_0.$$

Now, $1-\kappa = 1/\gamma$ so we get

$$\ln\theta = -\ln\rho + (1/\gamma)\ln p - \ln R' + \kappa\ln p_0.$$

Take d/dz to get

$$\frac{d\ln\theta}{dz} = -\frac{d\ln\rho}{dz} + (1/\gamma)\frac{d\ln p}{dz}.$$

From the hydrostatic equation we have

$$\frac{d\ln p}{dz} = -\frac{\rho g}{p} = -\frac{g}{R'T}$$

so therefore

$$\frac{d\ln\theta}{dz} = -\frac{d\ln\rho}{dz} - \frac{g}{\gamma R'T} = -\frac{d\ln\rho}{dz} - \frac{g}{c_s^2}.$$

3. a. Show that the group velocity for internal waves is

$$\vec{c}_g = \pm \frac{m^2 N}{\mathrm{K}_H \mathrm{K}^3}\left(k\hat{i} + l\,\hat{j} - \frac{\mathrm{K}_H^2}{m}\hat{k} \right).$$

Answer: Start with

$$\omega = \pm \frac{\mathrm{K}_H N}{\mathrm{K}}.$$

$$\frac{\partial \omega}{\partial k} = \pm \frac{N}{\mathrm{K}^2}\left(\mathrm{K}\frac{\partial \mathrm{K}_H}{\partial k} - \mathrm{K}_H \frac{\partial \mathrm{K}}{\partial k} \right) = \pm \frac{N}{\mathrm{K}^2}\left(\mathrm{K}\frac{k}{\mathrm{K}_H} - \mathrm{K}_H \frac{k}{\mathrm{K}} \right)$$

$$= \pm \frac{Nk}{\mathrm{K}^2}\left(\frac{\mathrm{K}^2 - \mathrm{K}_H^2}{\mathrm{K}_H \mathrm{K}} \right) = \pm \frac{Nm^2 k}{\mathrm{K}_H \mathrm{K}}.$$

In a similar manner we find

$$\frac{\partial \omega}{\partial l} = \pm \frac{Nm^2 l}{\mathrm{K}_H \mathrm{K}}; \quad \frac{\partial \omega}{\partial m} = \mp \frac{N\mathrm{K}_H m}{\mathrm{K}^3}$$

(note change of sign for $\partial\omega/\partial m$).

b. What is the magnitude of the group velocity for purely vertically propagating waves?

Answer: zero

c. What is the magnitude of the group velocity for purely horizontally propagating waves?

Answer: zero