ESCI 343 – Atmospheric Dynamics II
Answers to Selected Exercises for Lesson 4 – Introduction to Waves

3. A wave is represented in complex notation as

$$u(x,t) = Ae^{i(kx-\omega t)}$$

where $A = 2-3i$. Show that this is equivalent to representing the wave as

$$u(x,t) = 2\cos(kx-\omega t) + 3\sin(kx-\omega t).$$

Answer: Let $\theta = kx - \omega t$. Then,

$$u = (2-3i)(\cos\theta + i\sin\theta) = 2\cos\theta + 2i\sin\theta - 3i\cos\theta + -3i^2\sin\theta$$

$$= 2\cos\theta + 3\sin\theta + i(2\sin\theta - 3\cos\theta)$$

and we're only interested in the real part, so

$$u = 2\cos\theta + 3\sin\theta$$

4. Find the phase difference between the following two waves,

$$u(x,t) = Ae^{i(kx-\omega t)}$$

$$v(x,t) = Be^{i(kx-\omega t)}$$

for the following values of A and B.

a. $A = 2+3i$; $B = -3+2i$ **Answer:** $A \propto iB$; 270°

b. $A = 2+3i$; $B = -2-3i$ **Answer:** $A \propto -B$; 180°

c. $A = 2+3i$; $B = 3-2i$ **Answer:** $A \propto -iB$; 90°

d. $A = 2+3i$; $B = 4+6i$ **Answer:** $A \propto B$; 0°

e. $A = 2+3i$; $B = 9-6i$ **Answer:** $A \propto -iB$; 90°

5. a. Let a wave be represented by

$$u(x) = e^{ikx}.$$

Show that u and du/dx are 270° out of phase.

Answer:

$$\frac{du}{dx} = iku \propto iu$$

b. Let a wave be represented by

$$u(x) = \cos kx.$$

Show that u and du/dx are 270° out of phase, which shows the consistency of representing sinusoids using complex notation.

Answer:

$\frac{du}{dx} = -k \sin kx$, and $\cos x$ and $-\sin x$ are 270° out of phase.

9. What is the physical meaning of a complex frequency? In other words, if ω has an imaginary part, what does this imply? Hint: Put $\omega = \omega_r + i\omega_i$ into

$$u = e^{i(kx - \omega t)}$$

and see what you get.

Answer: $u = e^{i(kx - \omega_r t)} e^{\omega_i t}$, so waves amplify or decay exponentially with time.

10. Show that $c_g = \frac{\partial \omega}{\partial \mathrm{K}}$.

Answer:

$$c_g^2 = \left(\frac{\partial \omega}{\partial k}\right)^2 + \left(\frac{\partial \omega}{\partial l}\right)^2 + \left(\frac{\partial \omega}{\partial m}\right)^2 = \left(\frac{\partial \omega}{\partial \mathrm{K}} \frac{\partial \mathrm{K}}{\partial k}\right)^2 + \left(\frac{\partial \omega}{\partial \mathrm{K}} \frac{\partial \mathrm{K}}{\partial l}\right)^2 \left(\frac{\partial \omega}{\partial \mathrm{K}} \frac{\partial \mathrm{K}}{\partial m}\right)^2$$

$$= \left(\frac{\partial \omega}{\partial \mathrm{K}}\right)^2 \left[\left(\frac{\partial \mathrm{K}}{\partial k}\right)^2 + \left(\frac{\partial \mathrm{K}}{\partial l}\right)^2 + \left(\frac{\partial \mathrm{K}}{\partial m}\right)^2\right] = \left(\frac{\partial \omega}{\partial \mathrm{K}}\right)^2 \left[\frac{k^2}{\mathrm{K}^2} + \frac{l^2}{\mathrm{K}^2} + \frac{m^2}{\mathrm{K}^2}\right] = \left(\frac{\partial \omega}{\partial \mathrm{K}}\right)^2$$