

ESCI 342 – Atmospheric Dynamics I
Lesson 14 – Potential Vorticity

Reference: *An Introduction to Dynamic Meteorology* (4th edition), Holton
Atmosphere-Ocean Dynamics, Gill

Reading: Holton, Sections 5.3 and 5.4

THE BAROTROPIC QUASI-GEOSTROPHIC VORTICITY EQUATION

- Using the incompressible continuity equation we can write the quasi-geostrophic vorticity equation as

$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g = \eta_g \frac{\partial w}{\partial z}. \quad (1)$$

- If we integrate this equation from the surface to the top of the atmosphere we have

$$\int_{z_0}^{z_T} \left(\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g \right) dz = \int_{z_0}^{z_T} (\zeta_g + f) \frac{\partial w}{\partial z} dz = \eta_g [w(z_T) - w(z_0)] \quad (2)$$

- In the case of a barotropic fluid the velocity (and hence, vorticity) is independent of height, so we get

$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g = \frac{\eta_g}{h} (w_T - w_0) = \frac{\eta_g}{h} \left(\frac{Dz_T}{Dt} - \frac{Dz_0}{Dt} \right). \quad (3)$$

- The right-hand-side can be rewritten as

$$\frac{Dz_T}{Dt} - \frac{Dz_0}{Dt} = \frac{D}{Dt} (z_T - z_0) = \frac{Dh}{Dt} \quad (4)$$

where h is the total depth of the fluid.

- Equation (3) can therefore be written as

$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla_H \zeta_g + \beta v_g = \frac{\eta_g}{h} \frac{Dh}{Dt}. \quad (5)$$

BAROTROPIC POTENTIAL VORTICITY

- Equation (5) can be rearranged to

$$\frac{D}{Dt} \ln(\zeta_g + f) = \frac{D}{Dt} \ln h$$

or

$$\frac{D}{Dt} \left(\frac{\zeta_g + f}{h} \right) = 0.$$

- The quantity $(\zeta_g + f)/h$ is conserved following the fluid parcel in a barotropic fluid.
 - This quantity is called the *barotropic potential vorticity*.

POTENTIAL VORTICITY IN A BAROCLINIC FLUID

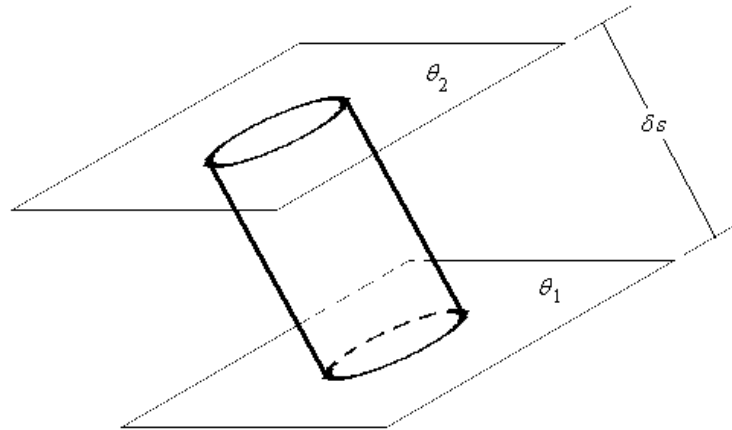
- The circulation theorem for a baroclinic fluid is

$$\frac{DC}{Dt} = -\oint \alpha dp = -\oint \frac{dp}{\rho}. \quad (6)$$

- The equation of state for the atmosphere can be written as

$$\rho = \frac{p}{R_d T} = \frac{p}{R_d \theta \left(\frac{p}{p_0}\right)^\kappa} = \frac{p_0^\kappa}{R_d \theta} p^{1-\kappa} \quad (7)$$

- On a surface of constant potential temperature (7) shows us that the density is a function of pressure only.¹ This makes the right-hand-side of (6) equal to zero. Thus:
 - **On an isentropic surface the circulation is constant.**
- Imagine a section of a stream tube of cross-sectional area δA lying between two isentropic surfaces.



- The circulation on the isentropic surface is given by

$$C = \omega \delta A.$$

The mass contained within the stream tube is

$$m = \rho \delta A \delta s$$

so that the circulation per unit mass is

$$\frac{C}{m} = \frac{\omega}{\rho \delta s}.$$

The length of the stream tube is

$$\delta s = \frac{\partial s}{\partial \theta} \delta \theta$$

so that the circulation per unit mass is

$$\frac{C}{m} = \frac{\omega}{\delta \theta \rho} \frac{\partial \theta}{\partial s}.$$

- If the flow is adiabatic then the ends of the stream tube will always lie on the isentropic surfaces. Thus, m is constant (as is $\delta \theta$). Since the circulation (C) is also constant, this means that

$$\frac{\omega}{\rho} \frac{\partial \theta}{\partial s} = \text{const.}$$

- On the synoptic scale we can assume that isentropic surfaces are nearly horizontal, so that the stream tube is oriented along the z -axis. Therefore we can write

¹ This result is actually true for any fluid (not just ideal gases) for which density can be written as a function of only p and θ .

$$\frac{\eta_\theta}{\rho} \frac{\partial \theta}{\partial z} = \text{const.},$$

and in terms of pressure this is

$$\frac{\eta_\theta}{\rho} \frac{\partial \theta}{\partial p} \frac{\partial p}{\partial z} = \text{const.}$$

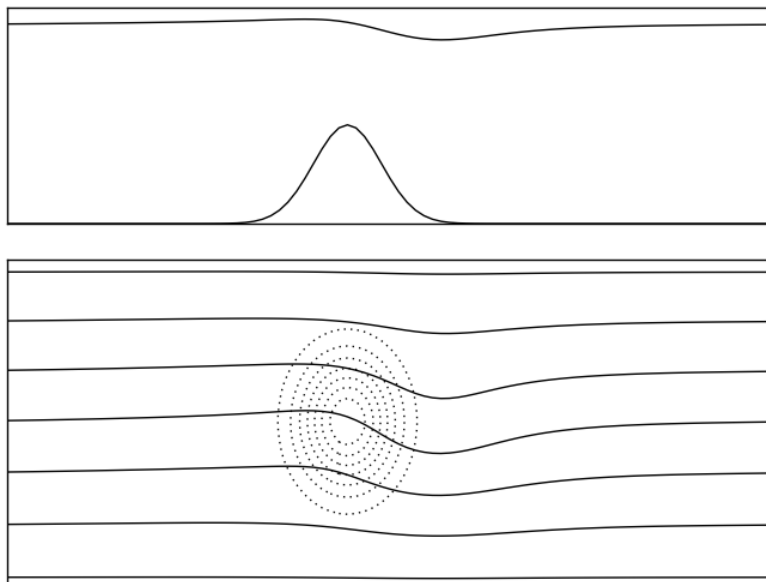
- If the atmosphere is hydrostatic then this becomes

$$-g \eta_\theta \frac{\partial \theta}{\partial p} = \text{const.}$$

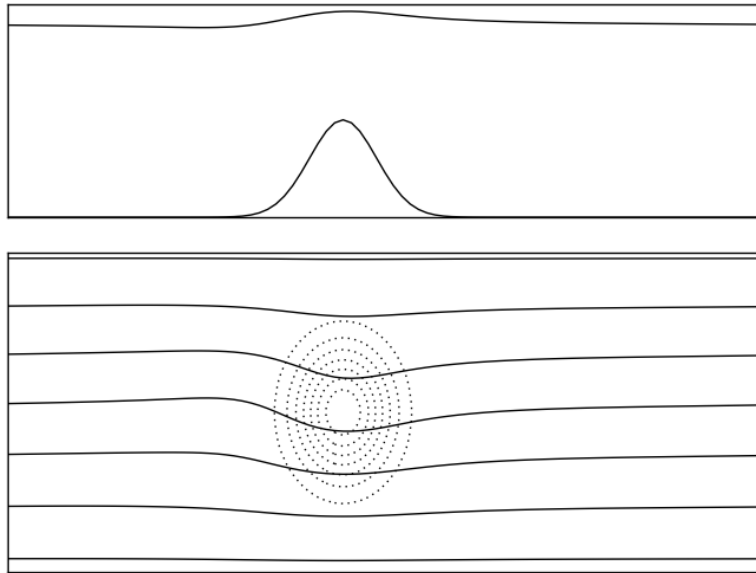
- This quantity is called *Ertel's potential vorticity*, and is the form of potential vorticity appropriate to the atmosphere. It is conserved following an air parcel in adiabatic flow, and is therefore a good tracer of air parcels under conditions where diabatic heating (latent heat of condensation, radiation, etc.) can be neglected.

BAROTROPIC POTENTIAL VORTICITY AND LEE-SIDE TROUGHS

- Conservation of potential vorticity can be used to explain the formation of lee-side troughs in westerly flow over a mountain barrier.
- The diagram below shows a model simulation for a barotropic fluid. The model was initialized with a purely westerly flow over a Gaussian shaped hill.
 - The upper diagram is a cross section along the x-axis for the middle of the domain, and shows the upper surface of the fluid and the model terrain
 - The lower diagram is a plot of the streamfunction.
- We can qualitatively explain the features we see as follows:



- o As the parcel starts to ascend the hill its depth decreases. This requires the absolute vorticity to also decrease. The atmosphere accomplishes this by creating both anticyclonic shear and curvature vorticity on the upstream side of the hill.
- o As the parcel crests the hill and starts moving back down slope, the depth starts to increase. The absolute vorticity now must increase, and so the atmosphere adjusts by having both cyclonic shear and curvature on the downstream side of the hill.
- o The result is a trough downstream (on the lee side) of the hill.
- For easterly flow a very different set of events occurs. This is shown in the diagram below.
- We explain the results qualitatively as:



- o As the parcel ascends the East side of the hill, again the absolute vorticity must decrease. This is accomplished by the parcel heading South, to where the planetary vorticity is less. However, the curvature cannot be too great here, because the curvature is cyclonic and would work against a decrease in absolute vorticity.
- o Once the parcel crests the hill and stretches as it descends, the absolute vorticity must once again increase, so the parcel curves gently back to the North.
- o In easterly flow, instead of a lee-side trough, a ridge is formed directly over the obstacle.

EXERCISES

1. Show that $w(z_T) - w(z_0) = Dh/Dt$
2. Expand $\frac{D}{Dt} \left(\frac{\zeta + f}{h} \right) = 0$ to show that it gives $\frac{\partial \zeta}{\partial t} + \vec{V} \cdot \nabla_h \zeta + \beta v = \frac{(\zeta + f) Dh}{h Dt}$.
3. Use the conservation of Ertel's potential vorticity to demonstrate the formation of a lee-side trough for westerly flow, and a ridge over the hill for easterly flow.