ESCI 342 – Atmospheric Dynamics I
Lesson 12 – Vorticity

Reference: *An Introduction to Dynamic Meteorology* (4^{rd} edition), Holton
An Informal Introduction to Theoretical Fluid Mechanics, Lighthill
Reading: Martin, Section 5.2

VORTICITY

- The circulation of a fluid is defined as

$$C \equiv \oint \vec{V} \bullet d\vec{l} \ . \qquad \textbf{(1)}$$

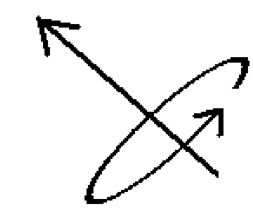
- From Stokes's theorem this is the same as

$$C \equiv \int_A \left(\nabla \times \vec{V}\right) \bullet d\vec{A} . \qquad \textbf{(2)}$$

  The quantity $\nabla \times \vec{V}$ is therefore also a measure of the rotation of the fluid, and is called the ***vorticity***.
  - Vorticity is defined as

$$\vec{\omega} \equiv \nabla \times \vec{V} \ . \qquad \textbf{(3)}$$

- ***<u>Vorticity is a vector</u>***. The rotation of the fluid follows the right-hand-rule with respect to the vorticity vector.



- Circulation and vorticity are closely related.
  - For a flat surface we can write (2) using the generalized mean-value theorem as

$$C \equiv \overline{\nabla \times \vec{V}} \bullet \int_A d\vec{A} = \overline{\nabla \times \vec{V}} \bullet \vec{A} \qquad \textbf{(4)}$$

    which is interpreted as meaning that the circulation around a planar surface is just the area-averaged vorticity normal to the surface multiplied by the area.
- The components of the vorticity vector are

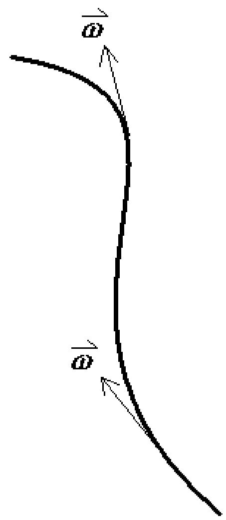
$$\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} \ . \qquad \textbf{(5)}$$

- ***In meteorology we are primarily concerned with circulations in the horizontal plane, so we are most interested in the vertical component of vorticity. From now on, when we speak of vorticity, we will usually be referring only to the vertical component.***
- The vertical component of vorticity is given the symbol ζ, and the following definition holds.

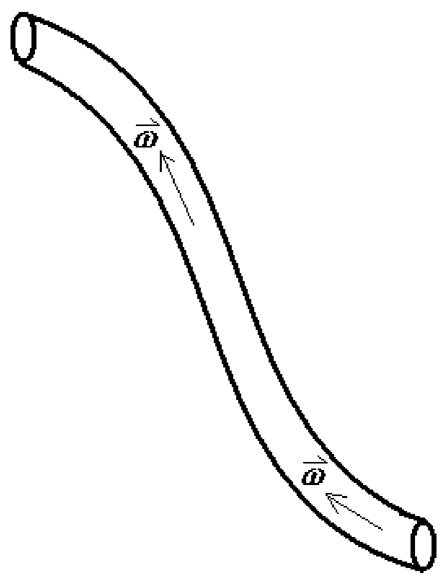
$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} . \qquad \textbf{(6)}$$

VORTEX STRETCHING

- A *vortex line* is a line that is everywhere parallel to the vorticity vector.



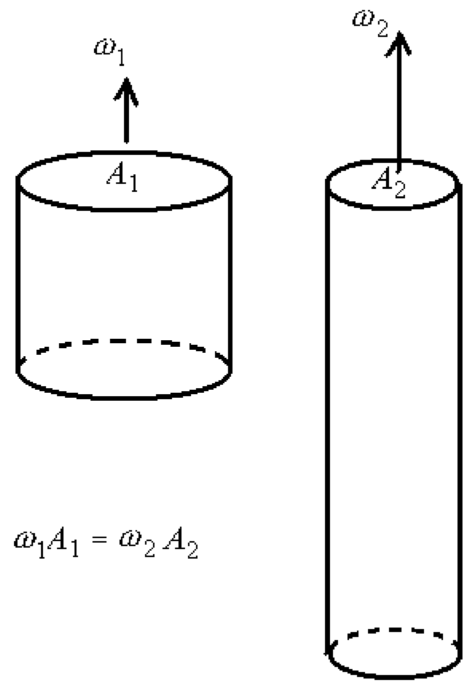
  - Vortex lines cannot begin or end in the interior of the fluid. They must terminate at a boundary of some sort.
  - Vortex lines move with the fluid.
- A *vortex tube* is a collection of vortex lines.



  - Vortex tubes move with the fluid and always consists of the same fluid parcels.
- The circulation taken around a disk that is perpendicular to the vortex tube is equal to the average vorticity of the tube times the area of the tube

$$C = \oint \vec{V} \bullet d\vec{l} = \int_A \left(\nabla \times \vec{V}\right) \bullet dA = \int_A \vec{\omega} \bullet d\vec{A} \equiv \overline{\omega} A \qquad \textbf{(7)}$$

- In a barotropic fluid, if a vortex tube is stretched the circulation doesn't change. However, the cross-sectional area of the tube will decrease, which means that the vorticity must increase!
  - Stretching a vortex tube causes it to spin faster.



- This phenomenon is known as *vortex stretching*.
  - Vortex stretching explains why a whirlpool forms over your bathtub drain. As the vortex tube moves over the drain it becomes stretched, causing it to spin more rapidly.
  - The sense of the rotation is determined by the original rotation of the vortex tube before it moved over the drain.
  - Vortex stretching also helps explain the formation of mesocyclones and tornadoes, as vertically oriented vortex tubes are stretched in the thunderstorm updraft.

RELATIVE VERSUS ABSOLUTE VORTICITY

- As with circulation, vorticity also depends on whether it is measured in an absolute reference frame or in a rotating frame.
  - The vorticity measured in the absolute reference frame is called *absolute vorticity*, and is given the symbol η.
  - The vorticity measured relative to the Earth is called *relative vorticity*, and is given the symbol ζ.
  - The vorticity of the surface of the Earth is called planetary vorticity. It is equal to the Coriolis parameter, *f* (see exercise 1).
- Absolute, relative, and planetary vorticity are related via

$$\eta = \zeta + f \,. \qquad \textbf{(8)}$$

- In the atmosphere, relative vorticity is usually much less than the planetary vorticity. Therefore, the absolute vorticity is usually a positive value.
  - ***Interesting aside:*** When we looked at the gradient wind back in Lesson 10 we saw that there were two anomalous cases of balanced flow: an anomalous high

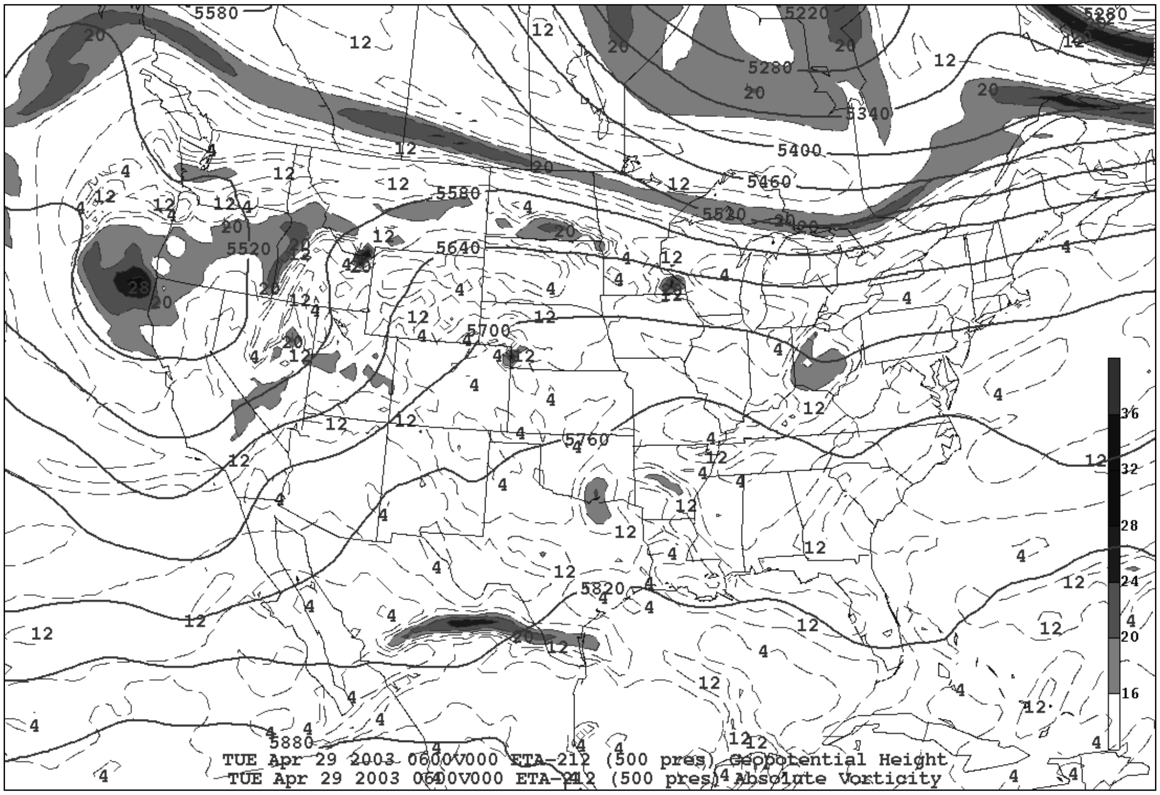
and an anomalous low. It turns out that both of these solutions have negative absolute vorticity, which may be one reason why they aren't observed frequently on the synoptic scale, since it is difficult to think of a process in the atmosphere that would generate negative absolute vorticity over a large area.

CURVATURE VERSUS SHEAR

- Vorticity may be visualized by imagining a paddle wheel moving with the fluid flow.
  - If the paddle wheel is rotating clockwise then there is negative (or anticyclonic) relative vorticity.
  - If the paddle wheel is rotating counter-clockwise then there is positive (or cyclonic) relative vorticity.
- The relative vorticity may be due either to
  - Curvature
  - Shear
- This is best visualized in natural coordinates, where the vorticity can be written as

$$\zeta = \underbrace{-\frac{\partial V}{\partial n}}_{A} + \underbrace{\frac{V}{R}}_{B} . \qquad \textbf{(9)}$$

  Note that *n* is always toward the left of the velocity vector. *Term A* is the vorticity due to ***shear***, and *Term B* is vorticity due to ***curvature***
  - You cannot necessarily tell the sense of vorticity by just looking at the streamlines. It is possible to have cyclonic curvature with anticyclonic shear, or vice-versa. In most cases, $\nabla \times \vec{V}$ must be calculated to find the sign of the relative vorticity.
  - Relative vorticity in the atmosphere is usually on the order of 10^{-5} to10^{-4} s^{-1}.
- The picture below shows the 500-mb geopotential height, and the 500-mb absolute vorticity (units are $s^{-1} \times 10^5$).



GEOSTROPHIC VORTICITY

- The vorticity due to the geostrophic wind is called the *geostrophic vorticity*, ζ_g.
- Since the geostrophic wind can be given in terms of a streamfunction,

$$u_g = -\frac{\partial \psi}{\partial y}; \qquad v_g = \frac{\partial \psi}{\partial x},$$

  the geostrophic vorticity is equal to the LaPlacian of the streamfunction,

$$\zeta_g = \nabla^2 \psi \,. \qquad \textbf{(10)}$$

  - Since the stream function is related to the geopotential field via

$$\psi = f_0^{-1}\Phi = f_0^{-1} g_0 Z$$

    (where $f = f_0$ = constant) then the geostrophic vorticity can be calculated directly from the geopotential heights.
- On the synoptic scale we often approximate the actual wind by the geostrophic wind. In the same vein, we often approximate the actual relative vorticity by the geostrophic vorticity.
  - This is convenient, since we can calculate vorticity directly from the geopotential heights, and don't need the actual wind observations.
- Remember…the geostrophic vorticity, like the geostrophic wind, is a definition. It is close to, but not necessarily equal to, the actual vorticity.

$$\zeta = \zeta_g + \zeta_a \cong \zeta_g \,.$$

EXERCISES

1. a. Show that the circulation of a flat disk of radius r in solid-body rotation is $C = 2\pi P r^2$ where P is the component of angular velocity perpendicular to the disk.

b. Show that the component of vorticity perpendicular to the disk is just $2P$.

c. Use your result to show that the vorticity of a point on the Earth's surface is $2\Omega \sin\phi$ and is therefore equal to the Coriolis parameter.

2. A vertically oriented vortex tube is in your bathtub. The tube is circular with a radius of 5 cm. The tube is rotating clockwise (as viewed from above) with a tangential velocity of 0.5 cm/s.

a. Calculate the average vorticity of the tube.

b. As the tube moves over the drain it is stretched, and its radius shrinks to 1 cm. What is the new average vorticity?

3. Calculate the vorticity of the following flows at point (x,y) = (1m, 2m).

a. $u = u_0 xy$, $v = v_0 y$ $\quad u_0 = 2\ \text{m}^{-1}\ \text{s}^{-1}$, $v_0 = 1\ \text{s}^{-1}$

b. $u = u_0 y$, $v = v_0 x$ $\quad u_0 = 2\ \text{s}^{-1}$, $v_0 = 1\ \text{s}^{-1}$

c. $u = u_0$, $v = v_0 x^2$ $\quad u_0 = 2\ \text{m s}^{-1}$, $v_0 = 1\ \text{m}^{-1}\ \text{s}^{-1}$

d. $u = u_0$, $v = v_0 \cos kx\ \sin ly$ $\quad u_0 = 2\ \text{m s}^{-1}$, $v_0 = 1\ \text{m s}^{-1}$, $k = 2.1\ \text{m}^{-1}$, $l = 0.9\ \text{m}^{-1}$

4. Show that if f and ρ are assumed constant, the geostrophic vorticity on a constant altitude surface is

$$\zeta_g = \frac{1}{f\rho}\nabla^2 p\,.$$

5. a. Show that if f is allowed to vary with latitude, that the geostrophic vorticity on a constant pressure surface is

$$\zeta_g = \frac{1}{f}\nabla^2\Phi - \frac{\beta}{f^2}\frac{\partial\Phi}{\partial y}\,.$$

b. Use scale analysis arguments to see if it is reasonable to ignore the second term in the above expression on the synoptic scale so that we can still use

$$\zeta_g \cong \frac{1}{f}\nabla^2\Phi$$

even though *f* is not constant with latitude. Hint: $f^{-1}\,\partial\Phi/\partial y$ is of the same order of magnitude as the geostrophic wind.