

ESCI 343 - Atmospheric Dynamics
Lesson 11 - Circulation
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References:

An Introduction to Atmospheric Dynamics, 4th ed., Holton
An Informal Introduction to Theoretical Fluid Mechanics, Lighthill

Barotropic and Baroclinic Fluids

- A *barotropic* fluid is one in which surfaces of constant pressure and constant density are parallel (see Fig. 1).

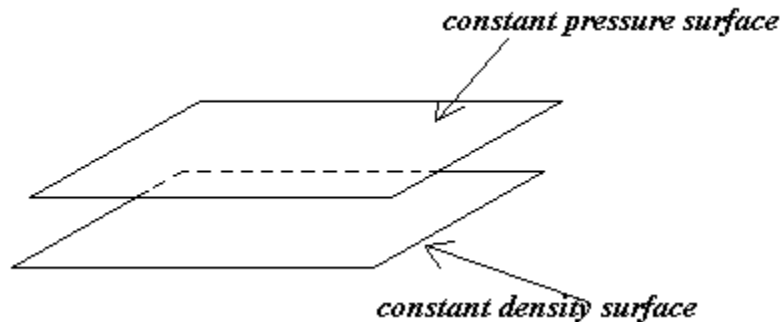


Figure 1: Pressure and density surfaces are parallel in a barotropic fluid.

- In a barotropic fluid the density is constant along a constant pressure surface.
 - From the ideal gas law this implies that, for a barotropic atmosphere, temperature is constant on a constant pressure surface, since $p/\rho = R_d T$, and since both p and ρ are constant on a pressure surface, then so would T .
 - If the atmosphere were truly barotropic there would be no isotherms on a constant pressure map.
- In a barotropic fluid the thermal wind is zero. Therefore, the flow is the same at all levels. ***There is no vertical wind shear in a barotropic atmosphere.***
- If a fluid is not barotropic it is *baroclinic*. In baroclinic fluids the pressure and density surfaces intersect as shown in Fig. 2.
 - In a baroclinic atmosphere there will be a temperature gradient on a constant pressure surface.
 - In a baroclinic atmosphere the flow will be different at different levels.

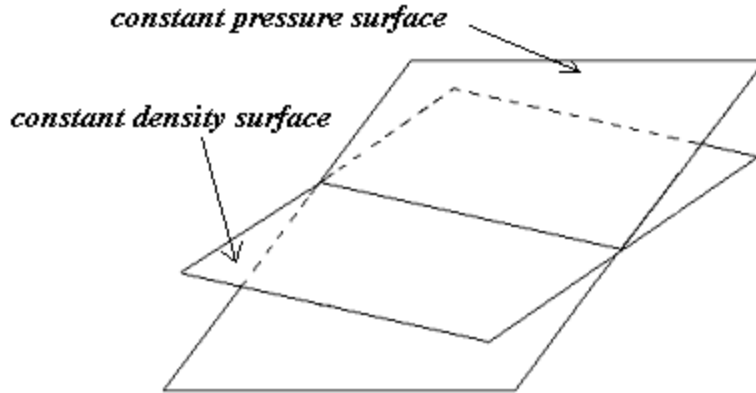


Figure 2: Pressure and density surfaces intersect in a baroclinic fluid.

- In the atmosphere the isotherms are sometimes parallel to the height contours (see Fig. 3).
 - In this case the wind changes speed with height, but is always in the same direction. Though technically baroclinic, this situation is typically referred to as *equivalent barotropic*.
 - **BEWARE!** Often times meteorologists say barotropic when they really mean equivalent barotropic.

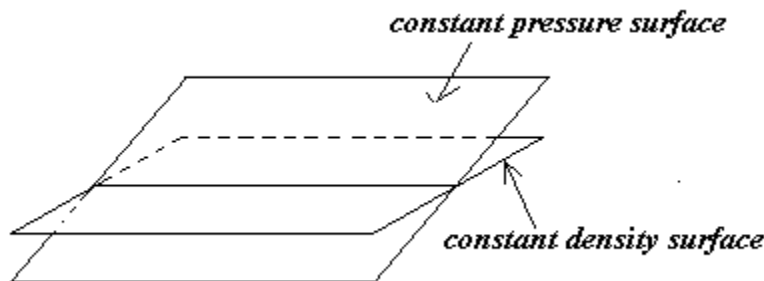


Figure 3: Pressure and density surfaces intersect along height contours in an equivalent barotropic fluid.

- The baroclinicity/barotropicity of the atmosphere can change with altitude.
 - Just because a particular pressure level is equivalent barotropic, does not mean the entire atmosphere above and below it is equivalent barotropic.
- To see if a particular level or region is barotropic, equivalent barotropic, or baroclinic we only have to look at a constant pressure surface and see the orientation of the isotherms (or thickness lines).

- If the isotherms are very widely spaced then the region or level is close to barotropic.
 - If the isotherms are parallel to the height contours then the region or level is equivalent barotropic.
 - If the isotherms cross the height contours the region or level is baroclinic.
- In Fig. 4 the solid lines are the 850 mb height contours and the dashed lines are the 500-1000 mb thickness contours.

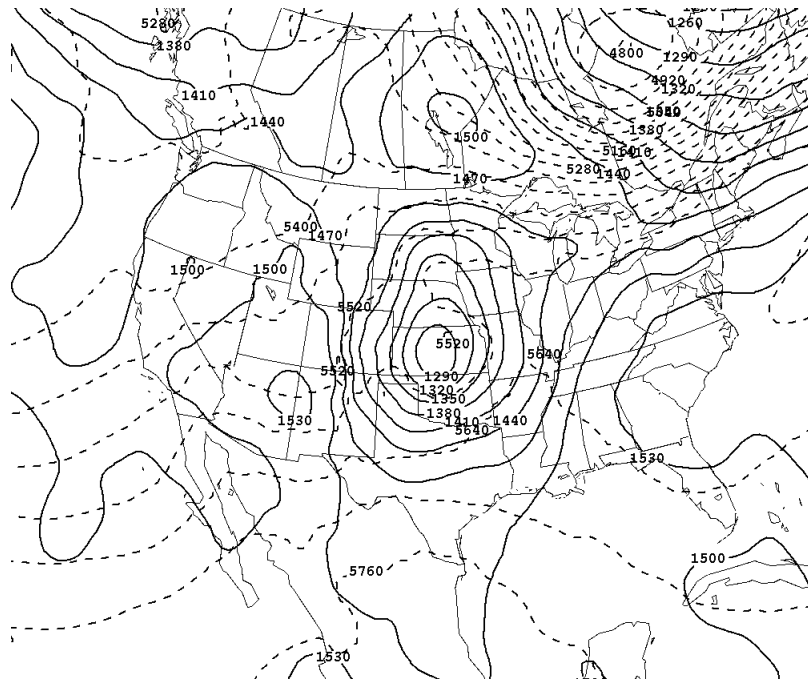


Figure 4: Example of 850 mb heights and 1000-500 mb thicknesses.

- The region near New England at 850 mb is highly baroclinic, since the height and thickness lines cross at large angles.
 - The low over Kansas at 850 mb is nearly equivalent barotropic since the height and thickness lines are nearly parallel.
- A map of the tropics (Fig. 5) shows that the height contours and thickness contours are spread very far apart (see example below). This is characteristic of the tropical atmosphere away from cyclones, etc. The tropics therefore tend to be barotropic (this is easy to remember since tropic appears in barotropic).
- Notice the equivalent barotropic low near the Philippines.
- A fluid that starts out barotropic can become baroclinic, except if the density is constant.
 - A fluid with constant density is called *autobarotropic*, because it is always barotropic (density is always constant on a constant pressure surface).

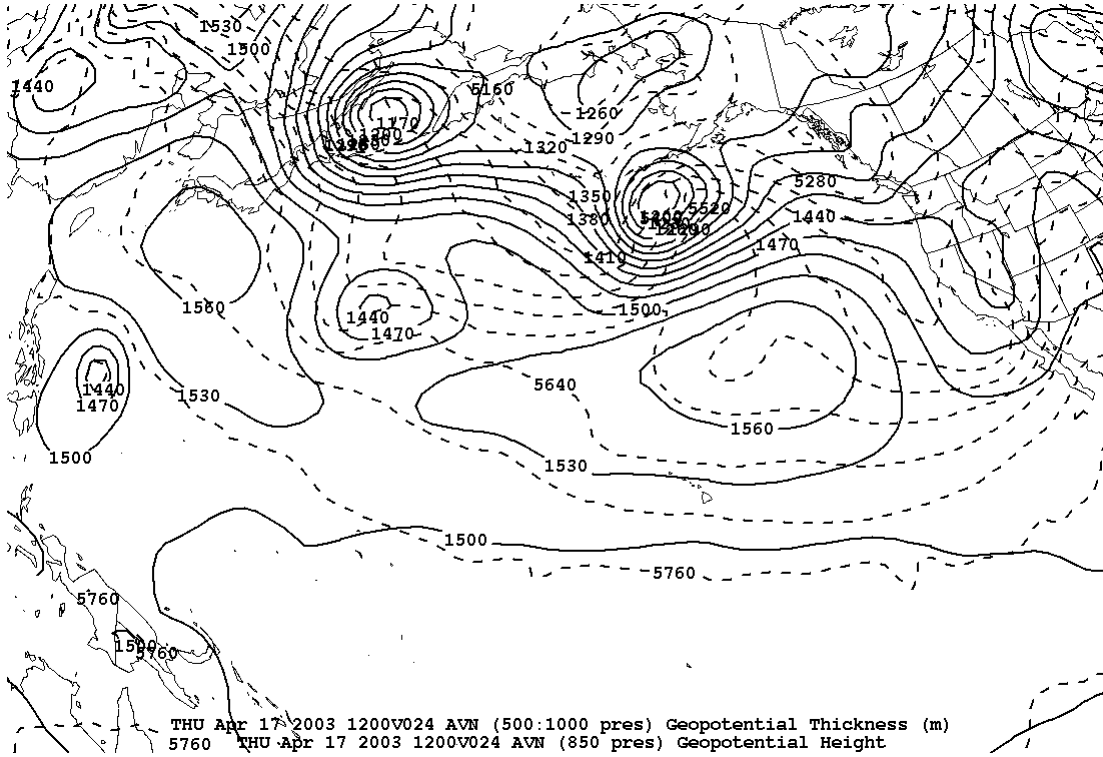


Figure 5: Example of 850 mb heights and 1000-500 mb thicknesses over the Tropical Pacific.

Mathematical Preliminaries

Here are a few mathematical preliminaries that will aid or upcoming discussion.

- **Stokes Theorem**, which states that a line integral of the tangent-component of a vector along a closed path can be expressed as the normal-component of the curl integrated over the area within the closed path,

$$\oint \vec{F} \cdot d\vec{l} = \int \int_A \nabla \times \vec{F} \cdot d\vec{A}. \quad (1)$$

- $d\vec{l}$ is a tangent vector along the closed path, and is positive in the counter-clockwise direction.
- The direction of $d\vec{A}$ is perpendicular to the surface bounded by the closed path, and is positive in the right-hand sense.
- The components of $d\vec{l}$ in Cartesian coordinates are

$$d\vec{l} = \hat{i}dx + \hat{j}dy + \hat{k}dz. \quad (2)$$

- The identity

$$df = \nabla f \cdot d\vec{l}, \quad (3)$$

where f is any scalar function.

- The identity

$$\nabla \times \nabla f = 0. \quad (4)$$

- **Exact differentials**, which are differentials whose integral around a closed path is zero,

$$\oint df = 0. \quad (5)$$

- If a function f of two variables x and y has a differential that is written as

$$df = Mdx + Ndy$$

then it is an exact differential if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad (6)$$

- If a differential is function of only a single variable and is of the form, $df = M(x)dx$, then it is an exact differential as long as M is integrable (f is differentiable).

- **Conservative vector fields**, which are vector fields \vec{H} that can be written in terms of the gradient of a scalar, such as

$$\vec{H} = \nabla s. \quad (7)$$

- A conservative vector field has the property

$$\oint \vec{H} \cdot d\vec{l} = 0, \quad (8)$$

which is shown as follows:

$$\oint \vec{H} \cdot d\vec{l} = \oint \nabla s \cdot d\vec{l} = \int \int_A \nabla \times \nabla s \cdot d\vec{A} = 0. \quad (9)$$

- So, if \vec{H} is a conservative vector field, then $\vec{H} \cdot d\vec{l}$ is an exact differential.
- Another important result from this is that for any scalar function, s , we have

$$\oint \nabla s \cdot d\vec{l} = 0. \quad (10)$$

Circulation

- Circulation is a measure of the rotation in a fluid.
- Circulation is defined as the line integral around a closed path of the dot-product of velocity and the vector tangent to the path,

$$C = \oint \vec{V} \cdot d\vec{l}. \quad (11)$$

- By convention, integrating around the path in a counter-clockwise direction is positive.
- Circulation will be most useful to us if we develop a formula for how it changes with time,

$$\frac{DC}{Dt} = \frac{D}{Dt} \oint \vec{V} \cdot d\vec{l} = \oint \frac{D}{Dt} (\vec{V} \cdot d\vec{l}) = \oint \frac{D\vec{V}}{Dt} \cdot d\vec{l} + \oint \vec{V} \cdot \frac{D}{Dt} d\vec{l}. \quad (12)$$

- The term

$$\oint \vec{V} \cdot \frac{D}{Dt} d\vec{l}$$

is evaluated as follows. We first note that

$$\frac{D}{Dt} d\vec{l} = d\vec{V}$$

(this result is not conceptually straight-forward, but is explained in the Appendix). So now we have

$$\oint \vec{V} \cdot \frac{D}{Dt} d\vec{l} = \oint \vec{V} \cdot d\vec{V} = \oint u du + \oint v dv + \oint w dw = \oint \frac{1}{2} du^2 + \oint \frac{1}{2} dv^2 + \oint \frac{1}{2} dw^2. \quad (13)$$

From (3) each of these terms can be written in the form of the integral of a gradient,

$$\oint \frac{1}{2} du^2 = \oint d \left(\frac{u^2}{2} \right) = \oint \nabla \left(\frac{u^2}{2} \right) \cdot d\vec{l}.$$

And then from (10) each of these terms is seen to be zero! Therefore,

$$\frac{DC}{Dt} = \oint \frac{D\vec{V}}{Dt} \cdot d\vec{l}. \quad (14)$$

Change in Circulation in an Absolute Reference Frame

- Before looking at the concept of circulation applied to the atmosphere, lets apply it to fluid in an absolute (non-rotating) coordinate system . In this case the momentum equation is

$$\frac{D\vec{V}}{Dt} = -\alpha \nabla p + \vec{g}^*, \quad (15)$$

where \vec{g}^* is the Newtonian gravitational acceleration, which can also be written as the gradient of a gravitational potential, $\vec{g}^* = -\nabla \Phi^*$.

- Inserting (15) into (14) yields,

$$\frac{DC}{Dt} = - \oint \alpha \nabla p \cdot d\vec{l} - \oint \nabla \Phi^* \cdot d\vec{l}. \quad (16)$$

- The last term in (16) has the form of (10), and is therefore zero, and so (16) becomes

$$\frac{DC}{Dt} = - \oint \alpha \nabla p \cdot d\vec{l} = - \oint \alpha dp \quad (17)$$

(remember that $\nabla p \cdot d\vec{l} = dp$).

- Equation (17) is the *Bjerknes Circulation Theorem*.
- Before exploring the significance of Bjerknes theorem for a baroclinic fluid, lets look at its application to a barotropic fluid.

Circulation in a Barotropic Fluid

- In a barotropic fluid the density is a function of pressure only. In this case we have

$$\oint \alpha dp = \oint f(p) dp = 0,$$

since $f(p)dp$ only involves a single variable, and is therefore an exact differential. So, for a barotropic fluid the circulation theorem, (17), becomes

$$\frac{DC}{Dt} = 0. \quad (18)$$

- This result is Kelvins circulation theorem.
- Kelvins circulation theorem states that the circulation around a closed curve moving with a frictionless, barotropic fluid is constant!
- Kelvins theorem is very powerful in that it expresses the dynamics of the fluid flow in one compact conservation law (conservation of circulation).
 - In general, physics problems are easier to solve if they can be written in terms of conservation laws. For example, it is easier to solve for the velocity of an object sliding down a frictionless ramp by using the conservation of energy rather than Newtons second law.
- Kelvins theorem (and Bjerknes for that matter) only apply to frictionless fluids.
 - Circulation can be created or dissipated in a boundary layer, due to friction at the surfaces, which creates velocity shear. This is not incorporated into either Kelvins or Bjerknes circulation theorem.

Solenoids

- We can write Bjerknæs' circulation theorem as

$$\frac{DC}{Dt} = - \oint \alpha dp = - \oint \alpha \nabla p \cdot d\vec{l} = - \int \int_A \nabla \times (\alpha \nabla p) \cdot d\vec{A}, \quad (19)$$

where Stokes' Theorem was used to get to the last step.

- The term $\nabla \times (\alpha \nabla p)$ evaluates as

$$\nabla \times (\alpha \nabla p) = \nabla \alpha \times \nabla p + \alpha \nabla \times \nabla p, \quad (20)$$

and by (4) the last term is zero. So, (19) becomes

$$\frac{DC}{Dt} = - \int \int_A (\nabla \alpha \times \nabla p) \cdot d\vec{A}. \quad (21)$$

- The physical meaning Bjerknæs Circulation Theorem is best illustrated if the path of integration in (21) lies in a plane, as shown in Fig. 6. In this case

$$\frac{DC}{Dt} = - \int \int_A |\nabla \alpha| |\nabla p| \sin \beta \, dA, \quad (22)$$

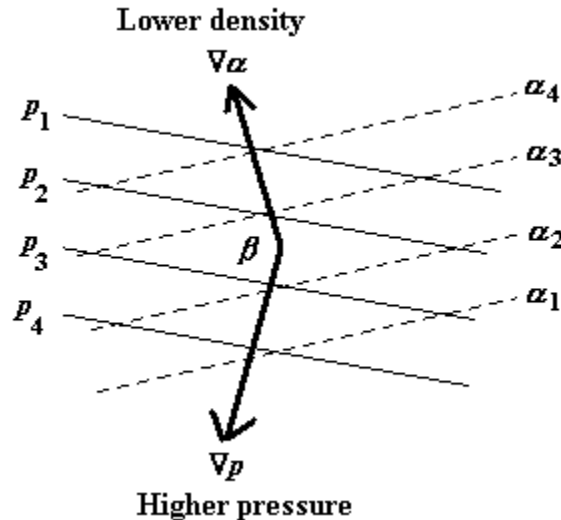


Figure 6: Solenoids of specific volume and pressure.

where β is the angle between the gradients of specific volume and pressure.

- In this example $DC/Dt < 0$, so a clockwise circulation would develop as shown in Fig. 7.

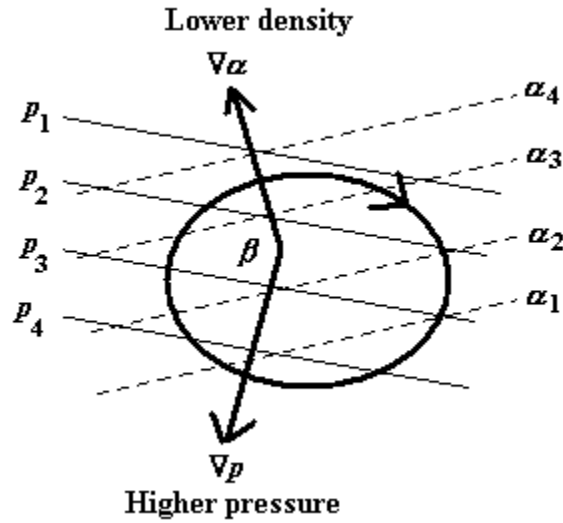


Figure 7: Solenoids resulting in a negative circulation tendency.

- In general the circulation that develops would be such that the density and pressure surfaces would become parallel.
- The diamond-shaped cells occurring between the isobars and isopycnals are called solenoids.
- For the atmosphere, which is an ideal gas, the solenoidal term can be written in terms of the temperature and pressure gradients as (see exercises)

$$\frac{DC}{Dt} = -R_d \int \int_A \nabla T \times \nabla(\ln p) \cdot d\vec{A}, \quad (23)$$

which is easier and more relevant to apply to atmospheric processes.

Circulation Theorem in a Rotating Reference Frame

- Up to now we have limited our discussion of circulation to an absolute reference frame (or to scales small enough that the rotation of the reference frame is negligible).
- To see why rotation of the frame makes a difference, imagine a ring (or chain) of fluid at rest with respect to the absolute frame of reference (see Fig. 8).
 - In a rotating reference frame there would appear to be a circulation oriented opposite to that of the rotation of the reference frame.
 - Alternatively, if the fluid were circulating in the absolute reference frame, but was at rest with respect to the rotating frame, the circulation with respect to the rotating frame would be zero.

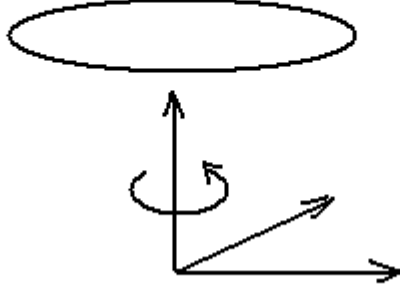


Figure 8: A ring or chain of fluid parcels in a rotating reference frame.

- To find the circulation in the rotating frame we need to use the momentum equation that includes both the Coriolis term and the centrifugal term. In this case the momentum equation is

$$\frac{D\vec{V}}{Dt} = -\alpha\nabla p - 2\vec{\Omega} \times \vec{V} + \vec{g}, \quad (24)$$

where \vec{g} is gravity (apparent gravity)

$$\vec{g} = \vec{g}^* + |\vec{\Omega}|^2 \vec{R},$$

and includes the centrifugal acceleration. Using (24) in (14) results in

$$\frac{DC}{Dt} = - \int \int_A (\nabla\alpha \times \nabla p) \cdot d\vec{A} - \oint (2\vec{\Omega} \times \vec{V}) \cdot d\vec{l} + \oint \vec{g} \cdot d\vec{l}, \quad (25)$$

where the steps to get the solenoidal term in (25) are the same as those used to derive the solenoidal term in (21).

- Since apparent gravity can be expressed in terms of the gradient of relative geopotential, $\vec{g} = -\nabla\Phi$, then by the same argument that we used to eliminate the gravitational (Newtonian gravity) acceleration in (16) we can eliminate the gravity (apparent gravity) acceleration term in (25), so we are left with

$$\frac{DC}{Dt} = - \int \int_A (\nabla\alpha \times \nabla p) \cdot d\vec{A} - 2 \oint (\vec{\Omega} \times \vec{V}) \cdot d\vec{l} \quad (26)$$

- From Stokes' Theorem

$$\oint (\vec{\Omega} \times \vec{V}) \cdot d\vec{l} = \int \int_A \nabla \times (\vec{\Omega} \times \vec{V}) \cdot d\vec{A}. \quad (27)$$

- The circulation theorem applied to the rotating frame is therefore

$$\frac{DC}{Dt} = \int \int_A (\nabla\alpha \times \nabla p) \cdot d\vec{A} - 2 \int \int_A \nabla \times (\vec{\Omega} \times \vec{V}) \cdot d\vec{A}. \quad (28)$$

Horizontal Circulations on the Earth

- For synoptic scale circulations we are primarily concerned with circulations around a vertical axis, so that $d\vec{A} = \hat{k}dA$. This greatly simplifies (28), which becomes

$$\frac{DC_z}{Dt} = \int \int_A \hat{k} \cdot (\nabla_H\alpha \times \nabla_H p) dA - 2 \int \int_A \hat{k} \cdot \nabla \times (\vec{\Omega} \times \vec{V}) dA. \quad (29)$$

- The subscript z on C_z is used to denote horizontal circulation (circulation around the vertical axis).
- The subscript H on the gradients in the solenoidal term indicate that we are only using the horizontal gradients of α and p .
- On the synoptic scale we can use the horizontal velocity $\vec{V} = u\hat{i} + v\hat{j}$. The angular velocity of the Earth in local Cartesian coordinates is $\vec{\Omega} = 0\hat{i} + \Omega \cos\varphi\hat{j} + \Omega \sin\varphi\hat{k}$. Using these vector components we can show that

$$2\hat{k} \cdot \nabla \times (\vec{\Omega} \times \vec{V}) = f\nabla_H \cdot \vec{V} + \frac{\partial f}{\partial y}v,$$

where $f = 2\Omega \sin\varphi$.

- Therefore, the equation for the change in circulation in the horizontal plane on a rotating Earth is

$$\frac{DC_z}{Dt} = - \int \int_A \hat{k} \cdot (\nabla_H\alpha \times \nabla_H p) dA - \int \int_A f(\nabla_H \cdot \vec{V}) dA - \int \int_A \beta v dA, \quad (30)$$

A
 B
 C

where $\beta = \partial f / \partial y$.

- The terms of this equation represent:

Term A: This is just the solenoidal term that we have seen before.

Term B: This is the divergence term.

- Divergence leads to anticyclonic circulation.
- Convergence leads to cyclonic circulation.

Term C: This is the β -effect term.

- Moving the chain of fluid parcels northward generates anticyclonic circulation.

– Moving the chain of fluid parcels southward generates cyclonic circulation.

- The solenoidal term can also be written in terms of the temperature and pressure gradients as (see exercises)

$$\frac{DC_z}{Dt} = -R_d \int_A \int_A \hat{k} \cdot (\nabla_H T \times \nabla_H \ln p) dA - \int_A \int_A f(\nabla \cdot \vec{V}) dA - \int_A \int_A \beta v dA. \quad (31)$$

- In this form, the circulation equations shows that if the isotherms are parallel to the isobars, then the solenoidal term is zero.
- This means that in an equivalent barotropic region of the atmosphere, the only way to develop a horizontal circulation is through divergence or north-south motion.

Appendix

- Figure 9 shows a segment of the closed path at two different times, t and $t + \Delta t$.

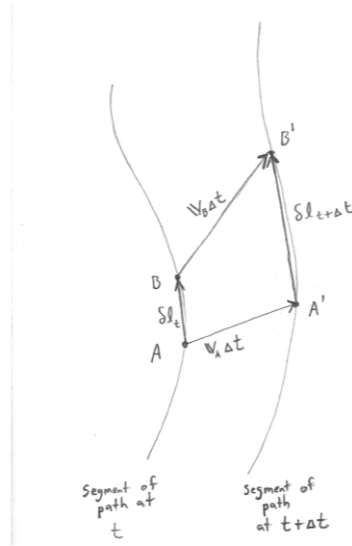


Figure 9: A segment of the fluid chain being advected.

- Point A is advected to Point A' , and Point B is advected to Point B' .
- The vectors $d\vec{l}_t$ and $d\vec{l}_{t+\Delta t}$ are very small, but finite.
- From the diagram we have

$$\vec{V}_A \Delta t + d\vec{l}_{t+\Delta t} = \vec{V}_B \Delta t + d\vec{l}_t,$$

which rearranges to

$$\frac{d\vec{l}_{t+\Delta t} - d\vec{l}_t}{\Delta t} = \vec{V}_B - \vec{V}_A = \delta \vec{V}.$$

- Taking the limit as $\Delta t \rightarrow 0$ results in

$$\frac{D}{Dt} \delta \vec{l} = \delta \vec{V}.$$

- If we allow the distance between Points A and B to approach zero, then $\delta \vec{l} \rightarrow d\vec{l}$ and $\delta \vec{V} \rightarrow d\vec{V}$, resulting in the desired result that

$$\frac{D}{Dt} d\vec{l} = d\vec{V}.$$

Exercises

- (a) Show that for an ideal gas that the solenoidal term of the circulation theorem can be written as

$$\frac{DC}{Dt} = - \oint \alpha dp = -R' \oint T d(\ln p).$$

- (b) Use identity (9) and Stokes Theorem to show that for an ideal gas the solenoidal term can be written as

$$\frac{DC}{Dt} = -R' \int \int_A \nabla T \times \nabla(\ln p) \cdot d\vec{A}.$$

- (a) A chain of fluid parcels in your bathtub lies in the horizontal plane. The chain is circular with a radius of 5 cm. The chain is rotating clockwise (as viewed from above) with a tangential velocity of 0.5 cm/s. As the chain moves over the drain, horizontal convergence causes the radius of the chain to shrink to 1 cm. What is the new tangential velocity? (Hint: Your bathwater is barotropic.)
 - Repeat part (a), only assume that initially the chain is rotating counter-clockwise.
 - What do you think determines the direction of the whirlpool that forms over your bathtub drain?

- Show that

$$\beta = \frac{2\Omega \cos \varphi}{a}$$

where a is the radius of the Earth.

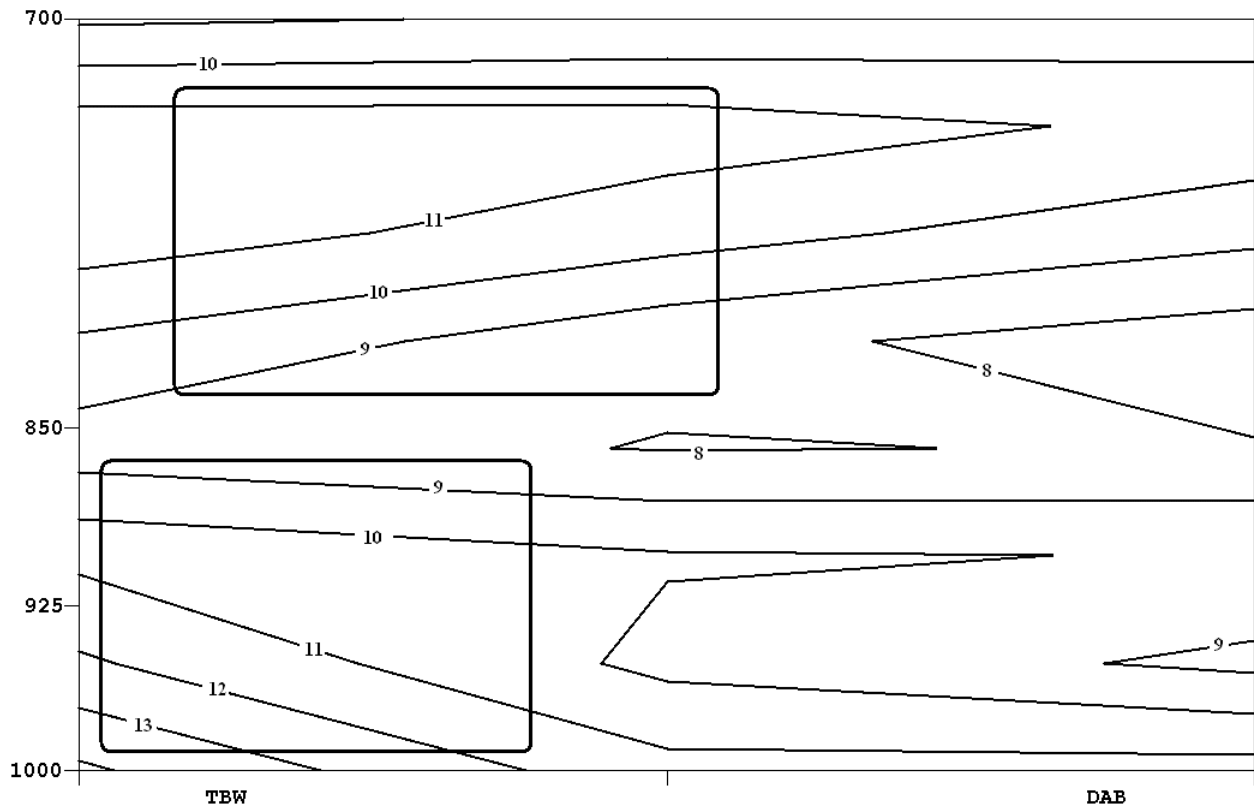
- A circular chain of fluid parcels with a radius of 300 km is centered at latitude 30N. Its circulation is initially zero. The entire chain of fluid parcels begins moving northward at 5 m/s.
 - What is the rate of change of the circulation?
 - Assuming β remains nearly constant with small changes in latitude, what will the circulation be after 24 hours?
 - How strong will the tangential winds be after 24 hours?

5. According to the circulation equation on a rotating Earth, a circular chain of fluid parcels initially at rest will generate an anticyclonic circulation as it expands, and a cyclonic circulation as it contracts. Give a physical explanation for why this occurs.

6. Show that

$$2\hat{k} \cdot \nabla \times (\vec{\Omega} \times \vec{V}) = f\nabla_H \cdot \vec{V} + \frac{\partial f}{\partial y}v.$$

7. The figure shows a vertical cross-section of temperature along a line across central Florida between Tampa Bay and Daytona Beach, Florida.



- (a) Draw arrows on the rectangles showing the orientation of the circulation that would develop due to the solenoidal term.
- (b) Would the circulation on the lower rectangle be a land breeze, or a sea breeze?