

ESCI 342 – Atmospheric Dynamics I
Lesson 7 – The Continuity and Additional Equations

Suggested Reading: Martin, Chapter 3

THE SYSTEM OF EQUATIONS IS INCOMPLETE

- The momentum equations in component form comprise a system of three equations with 4 unknown quantities (u , v , p , and ρ).

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \quad (1)$$

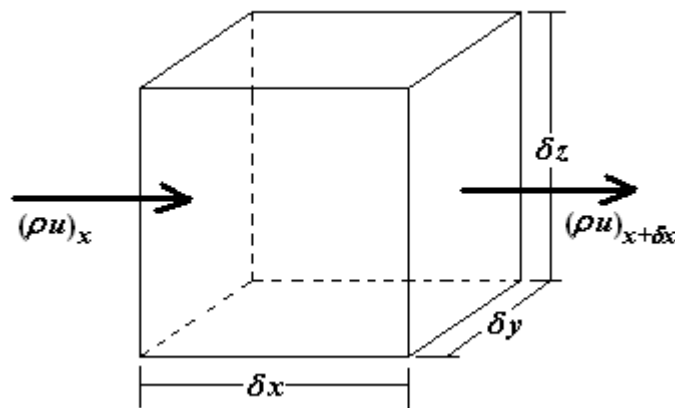
$$\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \quad (2)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad (3)$$

- They are not a closed set, because there are four dependent variables (u , v , p , and ρ), but only three equations. We need to come up with some more equations in order to close the set.

DERIVATION OF THE CONTINUITY EQUATION

- Another principle on which we can derive a new equation is the *conservation of mass*. The equation derived from this principle is called the *mass continuity equation*, or simply the *continuity equation*.
- Imagine a cube at a fixed point in space. The net change in mass contained within the cube is found by adding up the mass fluxes entering and leaving through each face of the cube.¹



- The mass flux across a face of the cube normal to the x -axis is given by ρu . Referring to the picture below, these fluxes will lead to a rate of change in mass within the cube given by

$$\frac{\partial m}{\partial t} = (\rho u)_x \delta y \delta z - (\rho u)_{x+\delta x} \delta y \delta z \quad (4)$$

¹ A flux is a quantity per unit area per unit time. Mass flux is therefore the rate at which mass moves across a unit area, and would have units of $\text{kg s}^{-1} \text{m}^{-2}$.

- The mass in the cube can be written in terms of the density as $m = \rho \delta x \delta y \delta z$ so that

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t} \delta x \delta y \delta z. \quad (5)$$

Equating (4) and (5) gives

$$\frac{\partial \rho}{\partial t} = \frac{(\rho u)_x - (\rho u)_{x+\delta x}}{\delta x} = - \left[\frac{(\rho u)_{x+\delta x} - (\rho u)_x}{\delta x} \right]$$

which becomes

$$\frac{\partial \rho}{\partial t} = - \frac{\partial(\rho u)}{\partial x}$$

as $\delta x \rightarrow 0$.

- Similar analysis can be done for the fluxes across the other four faces to yield the continuity equation,

$$\frac{\partial \rho}{\partial t} = - \frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z},$$

which can also be written in vector form as

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \vec{V}). \quad (6)$$

- Equation (6) is the *Eulerian form* of the continuity equation.
- A physical interpretation of (6) is that the change in density at a fixed point in space is dependent upon the divergence of the mass flux.
 - If there is divergence of the mass flux then $\nabla \cdot (\rho \vec{V}) > 0$ and density will decrease.
 - If there is convergence of the mass flux then $\nabla \cdot (\rho \vec{V}) < 0$ and density will increase.
- Using the vector identity

$$\nabla \cdot (\rho \vec{V}) = \rho \nabla \cdot \vec{V} + \vec{V} \cdot \nabla \rho \quad (7)$$

equation (6) can also be written in *Lagrangian form* as

$$\frac{D\rho}{Dt} = - \rho \nabla \cdot \vec{V}. \quad (8)$$

- Equations (6) and (8) are identical! You should be familiar with both forms of the continuity equation.
- A physical interpretation of (8) is that the change in density following a fluid parcel is dependent upon the velocity divergence.
 - If there is velocity divergence then $\nabla \cdot \vec{V} > 0$ and density will decrease.
 - If there is velocity convergence then $\nabla \cdot \vec{V} < 0$ and density will increase.

THE INCOMPRESSIBLE CONTINUITY EQUATION

- The Lagrangian form of the continuity equation is

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{V}. \quad (9)$$

- Under certain conditions the total derivative of pressure on the left-hand-side of the equation is much smaller than the right-hand-side, and we can ignore the time derivative. In this case the continuity equation is simply

$$\nabla \cdot \vec{V} = 0. \quad (10)$$

This is known as the *incompressible* continuity equation, because it is the form of the continuity equations obeyed by an incompressible fluid.

o Physically, *incompressibility means that the density of an air parcel doesn't change.*

- The conditions that must be met in order to use the incompressible continuity equation are (derivations are in the Appendix at the end of the lesson):
 - o **Condition A:** $U^2 \ll c^2$ where U is the flow speed and c is the speed of sound.²
 - o **Condition B:** $H \ll H_p$ where H is the vertical length scale of the flow, and H_p is the pressure scale height of the atmosphere.
- **Condition A** states that the flow speed must be much less than the speed of sound.
- **Condition B** states that the flow must be shallow compared to the scale height of the atmosphere.
- *Both Conditions A and B must hold in order for use of the incompressible continuity equation to be valid.*
- Though **Condition A** is met in the atmosphere, **Condition B** is not (except for very shallow circulations). Therefore, the incompressible continuity equation is not appropriate under most circumstances, because as an air parcel moves up and down in the atmosphere its density will change.

THE ANELASTIC CONTINUITY EQUATION

- If **Condition A** is met, but not **Condition B**, we can still come up with a simplified continuity equation for the atmosphere if we write the density in terms of a reference density that only changes with height, and a perturbation density that can change in any direction and with time such that

$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t)$$

where $\rho' \ll \rho$ (as is true in synoptic scale motion).

- Substituting this into the continuity equation gives

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{V}) + \nabla \cdot (\rho' \vec{V}) = 0$$

which scales as

$$\frac{\rho' U}{L} + \rho_0 \left(\frac{U}{L} + \frac{W}{H} \right) + \rho' \left(\frac{U}{L} + \frac{W}{H} \right) = 0.$$

Since $\rho' \ll \rho$ we can ignore the terms involving ρ' , so that we get the *anelastic* continuity equation

$$\nabla \cdot (\rho_0 \vec{V}) = 0. \quad (11)$$

² Laboratory experiments indicate that $U/C < \sim 0.5$ is sufficient.

- The anelastic continuity equation allows density changes due to vertical motion only.
- The anelastic continuity equation is the appropriate form of the continuity equation to use for the real atmosphere on the synoptic scale. However, for simplicity we will often make use the incompressible continuity equation instead (without introducing significant error for our purposes).

THERMODYNAMIC ENERGY EQUATION

- With the addition of the continuity equation we are up to 4 equations, but now we have 5 unknowns (u , v , w , p , and ρ). We need another equation. This is supplied by the thermodynamic energy equation.
- The thermodynamic energy equation comes from the 1st Law of Thermodynamics (conservation of energy). This equation is derived in detail in other courses, so it won't be derived here.

$$c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} = J \quad (12)$$

A B C

- This equation relates changes in temperature following an air parcel to adiabatic compression and expansion of the air parcel and to diabatic heating and cooling.
 - *Term A* is the temperature change following an air parcel.
 - *Term B* is the adiabatic heating and cooling term due to vertical motion.
 - *Term C* is the diabatic heating rate due to radiation, condensation, etc.
- If the motion is adiabatic, then $J = 0$.

THE EQUATION OF STATE

- We are now up to 5 equations, but the thermodynamic energy equation has introduced yet another variable, T . So we *still* need another equation. This is supplied by the equation of state (ideal gas law).
- The ideal gas law for air is

$$p = \rho R_d T (1 + 0.61q) \quad (13)$$

where q is the specific humidity.

THE WATER-MASS CONTINUITY EQUATION

- Equations (1), (2), (3), (8), (12), and (13) are six equations, but there are now seven unknowns! So, we still need another equation that hopefully doesn't introduce a new unknown.
- This new equation is the *water-mass continuity equation*,

$$\frac{\partial(\rho q)}{\partial t} + \nabla \cdot (\rho q \vec{V}) = S \quad (14)$$

where the term S takes into account the sources and sinks of water to the atmosphere.

THE GOVERNING EQUATIONS

- We've now derived the set of equations that govern the atmosphere on the synoptic scale. They are:

$$\frac{\partial u}{\partial t} + \vec{V} \cdot \nabla u = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} + \vec{V} \cdot \nabla v = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0 \quad \text{or} \quad \nabla \cdot (\rho \vec{V}) = 0 \quad \text{or} \quad \nabla \cdot \vec{V} = 0$$

$$c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} = J$$

$$\frac{\partial(\rho q)}{\partial t} + \nabla \cdot (\rho q \vec{V}) = \text{Sources and Sinks}$$

$$p = \rho R_d T (1 + 0.61q)$$

- **These constitute a closed set of 7 equations in 7 unknowns** ($u, v, w, p, T, q,$ and ρ).
- This set of equations is known as the governing equations. Theoretically, they can be solved to predict or diagnose the future values of the 7 variables.

APPENDIX – This section shows the derivations of conditions A and B. It is included for completeness, but will not be included on examinations or quizzes.

- The speed of sound in a fluid is given by the partial derivative of pressure with respect to density at constant entropy (potential temperature),

$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_\theta$$

- For an ideal gas, $c = \sqrt{\gamma R'T}$ where $\gamma = c_p / c_v$.
- The thermodynamic variables commonly used are $T, \rho, \theta,$ and p . We only need to specify two of them, and any others can be deduced from these two.
- We can therefore write density as a function of pressure and potential temperature, $\rho = f(p, \theta)$.
- The potential temperature (θ) is conserved under adiabatic conditions. Therefore, under adiabatic conditions $\rho = f(p)$, and

$$\frac{D\rho}{Dt} = \left(\frac{\partial \rho}{\partial p} \right)_\theta \frac{Dp}{Dt} = \frac{1}{c^2} \frac{Dp}{Dt}$$

This means the continuity equation can be written as

$$\frac{1}{\rho c^2} \frac{Dp}{Dt} = -\nabla \cdot \vec{V}$$

which can be expanded to

$$\frac{1}{\rho c^2} \left(\frac{\partial p}{\partial t} + \vec{V} \cdot \nabla p \right) = -\nabla \cdot \vec{V} \quad (15)$$

- o From the momentum equation

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla p - \hat{k} \times f \vec{V} + \vec{g}$$

we can solve for ∇p to get

$$\nabla p = \rho \frac{D\vec{V}}{Dt} - \rho \hat{k} \times f \vec{V} + \rho \vec{g}.$$

Substituting this into (15) gives

$$\frac{1}{\rho c^2} \left[\frac{\partial p}{\partial t} + \vec{V} \cdot \left(\rho \frac{D\vec{V}}{Dt} - \rho \hat{k} \times f \vec{V} + \rho \vec{g} \right) \right] = -\nabla \cdot \vec{V}$$

which reduces to

$$\frac{1}{c^2} \left(\frac{1}{\rho} \frac{\partial p}{\partial t} + \frac{1}{2} \frac{D|V|^2}{Dt} - gW \right) = -\nabla \cdot \vec{V} \quad (16)$$

- In terms of order of magnitude, this equation is

$$\frac{\delta P U}{\rho c^2 L} + \frac{U^3}{c^2 L} + \frac{gW}{c^2} = \frac{U}{L}.$$

- If the pressure suddenly changed at a point in the fluid by an amount δP , you would expect that the change in velocity would be given by

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla(\delta p)$$

and so δP would be of the order of ρU^2 . Therefore, the orders of magnitude of equation (16) become

$$\frac{U^3}{c^2 L} + \frac{U^3}{c^2 L} + \frac{gW}{c^2} = \frac{U}{L}.$$

- The first two terms of equation (16) can be ignored only if

$$\frac{U^3}{c^2 L} \ll \frac{U}{L},$$

which becomes

$$\frac{U^2}{c^2} \ll 1. \quad (\text{A})$$

- The third term can be ignored if

$$\frac{gW}{c^2} \ll \frac{U}{L},$$

which can also be written as

$$\frac{gH}{c^2} \frac{W}{H} \ll \frac{U}{L}.$$

Since

$$\frac{W}{H} \leq \frac{U}{L}$$

the condition becomes

$$\frac{gH}{c^2} \ll 1.$$

This can be written as

$$\frac{gH}{\gamma R'T} \cong \frac{1}{\gamma} \frac{H}{H_p} \ll 1 \quad \text{(B)}$$

where H_p is the pressure scale height.

EXERCISES

1. Show that ρu has the units of mass flux.
2. Use the vector identity $\nabla \cdot s\vec{A} = s\nabla \cdot \vec{A} + \vec{A} \cdot \nabla s$ to show that the two forms of the continuity equation we derived are equivalent.
3. You are studying the land-sea breeze circulation. This circulation has a typical depth of 1000 meters or so. Is it appropriate to use the incompressible continuity equation in this case?
4. Show that ρq is equal to absolute humidity, ρ_v .