

ESCI 342 – Atmospheric Dynamics I
Lesson 6 – Scale Analysis

SCALE ANALYSIS OF THE MOMENTUM EQUATIONS

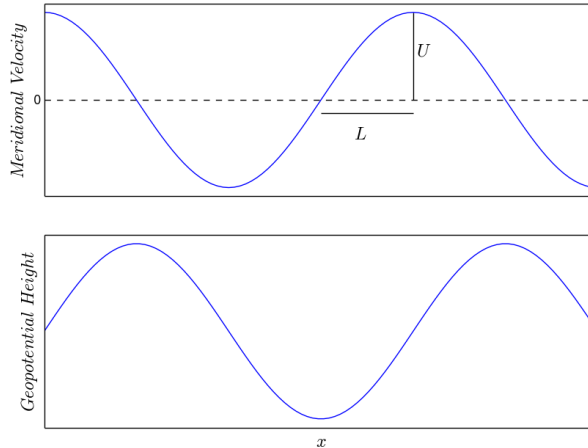
- Not all of the terms in the momentum equations are significant. If a term is much smaller than the others then it is reasonable to ignore it under certain circumstances.
- To assess which terms can be neglected, we assign an “order of magnitude” to all the variables and parameters in the equations.
 - For scale analysis we often don’t assign exact numbers... just orders of magnitude.
- The orders of magnitude are assigned for specific scales of motion. For instance, they would be quite different for the study of tornadoes than they would be for the study of hurricanes.
- The parameters that need to be scaled are shown in the table below:

Name	Symbol
Horizontal velocity	U
Vertical velocity	W
Horizontal length scale	L
Vertical length scale	H
Pressure change	δP
Density	ρ
Time ¹	$\tau = L/U$

- Appropriate scales for each are determined as follows:
 - Horizontal velocity, U : For most atmospheric circulations the u and v components are of similar magnitude, and so we use a single scale parameter, U , to represent both.
 - Vertical velocity, W .
 - Horizontal length scale, L : The horizontal length scale can be defined in a few ways.
 - For wavelike features in the atmosphere it is usually taken to be one-fourth of the total wavelength, $L \approx \lambda/4$.² This is because the scale of the spatial velocity derivatives such as $\partial u/\partial x$ are of the order of U/L (see figure below).
 - For a vortex, L is taken to be the radius R (not diameter), since the spatial derivative of velocity in a vortex will be of the order of U/R .

¹ The time scale $\tau = L/U$ is called the *advective* time scale. It is the time it would take for a parcel of fluid to travel the entire horizontal length of the flow.

² In very early work (Charney, J.G: ‘On the scale of atmospheric motions’, *Geof. Publ.*, **17**, 3-17; Burger, A.P., 1957: ‘Scale consideration of planetary motions of the atmosphere’, *Tellus*, **10**, 195-205) a horizontal scale of $\lambda/2$ was postulated. This was altered to $\lambda/4$ in Phillips, N.A., 1963: ‘Geostrophic motion’, *Rev. Geophys.*, **1**, 123-175



- o Vertical length scale, H : The vertical length scale is the height of the circulation or disturbance.
- o Pressure change, δP : The pressure change is needed for terms involving derivatives of the pressure.
 - In the horizontal, this will be the range between the maximum and minimum pressures found moving horizontally across the circulation.
 - In the vertical, it will be the maximum and minimum pressures found moving vertically through the circulation.
 - There may be very large differences for δp in the horizontal versus in the vertical.
- o Time, τ : For the time scale we use the *advective time scale*, defined as $\tau = L/U$. This is the time it would take for a parcel of fluid traveling at speed U to travel the distance L .

SYNOPTIC SCALE ANALYSIS OF THE HORIZONTAL MOMENTUM EQUATIONS

- For synoptic scales the following orders of magnitude are appropriate:

Name	Symbol	Order of magnitude
Horizontal velocity	U	10 m s^{-1}
Vertical velocity	W	0.01 m s^{-1}
Horizontal distance	L	$1000 \text{ km } (10^6 \text{ m})$
Vertical distance	H	$10 \text{ km } (10^4 \text{ m})$
Pressure change	δP	<i>Horizontal:</i> $10 \text{ mb } (10^3 \text{ Pa})$ <i>Vertical:</i> $1000 \text{ mb } (10^5 \text{ Pa})$
Time	$\tau = L/U$	$1 \text{ day } (10^5 \text{ s})$

- o The following parameters are also used:

density	ρ	1 kg m^{-3}
kinematic viscosity	ν	$1.46 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$
omega	Ω	$7.292 \times 10^{-5} \text{ rad s}^{-1}$
latitude	ϕ	45°
radius of Earth	a	$6.378 \times 10^6 \text{ m}$

- Using these scales and parameters, the terms in the u -momentum equation have the following orders of magnitude

$\frac{\partial u}{\partial t}$	$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$	$w \frac{\partial u}{\partial z}$	$-\frac{uv \tan \phi}{a}$	$\frac{uw}{a}$	$-\frac{1}{\rho} \frac{\partial p}{\partial x}$	$2\Omega v \sin \phi$	$-2\Omega w \cos \phi$
U^2/L	U^2/L	WU/H	U^2/a	UW/a	$\delta P/(\rho L)$	$2\Omega U \sin 45$	$2\Omega W \cos 45$
10^{-4}	10^{-4}	10^{-5}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}

$\nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$	$\nu \frac{\partial^2 u}{\partial z^2}$	$-\nu \frac{u}{a^2} (\tan^2 \phi + 1)$	$-\nu \frac{2}{a} \frac{\partial v}{\partial x} \tan \phi$	$\nu \frac{2}{a} \frac{\partial w}{\partial x}$
$\nu U/L^2$	$\nu U/H^2$	$\nu U/a^2$	$\nu U/La$	$\nu W/La$
10^{-16}	10^{-12}	10^{-18}	10^{-18}	10^{-20}

- A similar analysis for the v -momentum equation is

$\frac{\partial v}{\partial t}$	$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}$	$w \frac{\partial v}{\partial z}$	$\frac{u^2 \tan \phi}{a}$	$\frac{vw}{a}$	$-\frac{1}{\rho} \frac{\partial p}{\partial y}$	$-2\Omega u \sin \phi$
U^2/L	U^2/L	WU/H	U^2/a	UW/a	$\delta P/(\rho L)$	$2\Omega U \sin 45$
10^{-4}	10^{-4}	10^{-5}	10^{-5}	10^{-8}	10^{-3}	10^{-3}

$\nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$	$\nu \frac{\partial^2 v}{\partial z^2}$	$-\nu \frac{v}{a^2} (\tan^2 \phi + 1)$	$\nu \frac{2}{a} \frac{\partial u}{\partial x} \tan \phi$	$\nu \frac{w}{a^2} \tan \phi$	$\nu \frac{2}{a} \frac{\partial w}{\partial y}$
$\nu U/L^2$	$\nu U/H^2$	$\nu U/a^2$	$\nu U/La$	$\nu W/a^2$	$\nu W/La$
10^{-16}	10^{-12}	10^{-18}	10^{-18}	10^{-21}	10^{-20}

- Many of the terms are very small compared to others, and can be ignored without significant loss of accuracy. We can therefore ignore the curvature terms, the viscous terms, and the Coriolis term that involves the vertical velocity.
- Ignoring these terms yields a much simpler version of the horizontal equations of motions:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi \quad (2)$$

Note: We could have also ignored the vertical advection terms, but it is not too much of an inconvenience to keep them.

- By defining the *Coriolis parameter* as

$$f = 2\Omega \sin \phi \quad (3)$$

the horizontal momentum equations assume the form

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \quad (4)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \quad (5)$$

- In vector form the horizontal momentum equation is

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla p - \hat{k} \times f \vec{V}. \quad (6)$$

In (6), all derivatives and vectors are horizontal,

$$\vec{V} = u \hat{i} + v \hat{j}$$

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} \quad .$$

- The total derivative terms in (4), (5), and (6) are known as the *inertial terms*. The terms on the right-hand-side are the *pressure gradient* and *Coriolis terms* respectively.

THE ROSSBY NUMBER

- Dividing the horizontal momentum equation, (6), through by fV we get

$$\frac{1}{fV} \frac{D\vec{V}}{Dt} = -\frac{1}{\rho fV} \nabla p - \hat{k} \times \frac{f\vec{V}}{fV}.$$

- Using the representative scales the order of magnitude of these terms are

$$\frac{U}{fL} = \frac{\delta P}{\rho fU} + 1$$

- The dimensionless combination U/fL is defined as the *Rossby number* (named for Gustav Rossby),

$$Ro \equiv U/fL \quad (7)$$

GEOSTROPHIC BALANCE (VERY SMALL ROSSBY NUMBER)

- When the Rossby number is much less than unity ($Ro \ll 1$), then the acceleration (inertial) term can be ignored and the only two terms left are the pressure gradient term and the Coriolis term, which must be nearly in balance.
 - This is known as *geostrophic balance*, and the velocity in this case is known as the *geostrophic wind*.

- o The momentum equation in this case reduces to

$$\hat{k} \times f \vec{V}_g = -\frac{1}{\rho} \nabla p$$

which is solved for the geostrophic wind to yield

$$\vec{V}_g = \frac{1}{f \rho} \hat{k} \times \nabla p, \quad (8)$$

with wind speed (magnitude) equal to

$$V_g = \frac{|\nabla p|}{f \rho}. \quad (9)$$

CYCLOSTROPHIC BALANCE (VERY LARGE ROSSBY NUMBER)

- When the Rossby number is much greater than unity ($Ro \gg 1$) then the Coriolis term can be ignored. In this instance the only terms that are left are the acceleration and the pressure gradient terms, and so the acceleration is a direct result of the pressure gradient force

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla p. \quad (10)$$

- o This type of balance is called *cyclostrophic*.
- o In cyclostrophic balance the pressure gradient acceleration is exactly that required for the centripetal acceleration, and so we have

$$\frac{V_c^2}{r} = \frac{|\nabla p|}{\rho}$$

or

$$V_c = \sqrt{\frac{r|\nabla p|}{\rho}} \quad (11)$$

where r is the radius of curvature of the flow.

GRADIENT BALANCE (ROSSBY NUMBER NEAR UNITY)

- If the Rossby number is of the order of unity ($Ro \sim 1$), then all three terms must be retained. This is known as *gradient balance*, and the wind in this case is known as the *gradient wind*. Details of gradient wind will be discussed in a future lesson.
- The following table summarized these results

<i>Ro</i>	Terms	Balance
$\ll 1$	pressure gradient and Coriolis	<i>geostrophic</i>
~ 1	acceleration, pressure gradient, and Coriolis	<i>gradient</i>
$\gg 1$	acceleration and pressure gradient	<i>cyclostrophic</i>

- For large-scale (synoptic scale) motion, the Rossby number is of the order

$$Ro \sim \frac{(10m/s)}{(10^{-4}s^{-1})(10^6m)} = 0.1,$$

which shows that on these scales the atmosphere is close to being in geostrophic balance. Hence, the actual wind should be close to the geostrophic wind.

INERTIAL FLOW (ROSSBY NUMBER EXACTLY EQUAL TO ONE)

- If the pressure gradient is exactly zero, then the inertial terms must exactly balance the Coriolis term.³
- The balance in this case is called *inertial balance*, with the speed given by

$$V_{in} = -f r. \quad (12)$$

- In inertial balance the flow is circular, with radius of $R = -V_{in} / f$.
 - Since by definition V is always positive, then R must be negative and so inertial flow is anticyclonic.
- The period of the inertial flow is found by dividing the circumference of the inertial circle by the speed,

$$\tau = \frac{-2\pi r}{V_{in}} = \frac{2\pi}{f}. \quad (13)$$

- The inertial period is shorter at higher latitudes, and is infinity at the Equator.

MORE ON THE GEOSTROPHIC WIND

- ***The geostrophic wind is a definition! On the synoptic scale the actual wind should be close to the geostrophic wind (because $Ro \ll 1$), but will rarely be exactly equal to the geostrophic wind.***
- The components of the geostrophic wind are

$$u_g = -\frac{1}{f \rho} \frac{\partial p}{\partial y} \quad (14)$$

$$v_g = \frac{1}{f \rho} \frac{\partial p}{\partial x} \quad (15)$$

- ***The geostrophic wind is parallel to the isobars with lower pressure to the left (in the Northern Hemisphere).***
- ***The geostrophic wind speed is directly proportional to the pressure gradient.***
- In pressure coordinates, the geostrophic wind and components are

$$\vec{V}_g = \frac{1}{f} \hat{k} \times \nabla \Phi = \frac{g_0}{f} \hat{k} \times \nabla Z \quad (16)$$

$$u_g = -\frac{1}{f} \frac{\partial \Phi}{\partial y} = -\frac{g_0}{f} \frac{\partial Z}{\partial y} \quad (17)$$

$$v_g = \frac{1}{f} \frac{\partial \Phi}{\partial x} = \frac{g_0}{f} \frac{\partial Z}{\partial x} \quad (18)$$

- Therefore, on a constant pressure surface
 - ***The geostrophic wind is parallel to the isohypses with lower heights to the left (in the Northern Hemisphere).***

³ In this instance the Rossby number would be exactly equal to one. However, a Rossby number of one does not automatically imply inertial balance, because when calculating Rossby number we use characteristic orders of magnitude for the flow parameters, not exact values. Only if the pressure gradient is exactly zero do we have inertial balance. Pure inertial flow rarely if ever occurs. However, there often is an inertial component to the flow. Inertial balance plays a role in some atmospheric phenomena such as nocturnal low-level jets.

- *The geostrophic wind speed is directly proportional to the geopotential height gradient.*
- Another important feature of the geostrophic wind is that it is non-divergent ($\nabla \cdot \vec{V}_g = 0$) if f is constant.
- The geostrophic wind is sometimes written in terms of the *streamfunction*, ψ , defined as $\psi \equiv p/f\rho$ (in pressure coordinates $\psi = \Phi/f$), where f and ρ are constant. In this case, the geostrophic wind is

$$\vec{V}_g = \hat{k} \times \nabla \psi \quad (19)$$

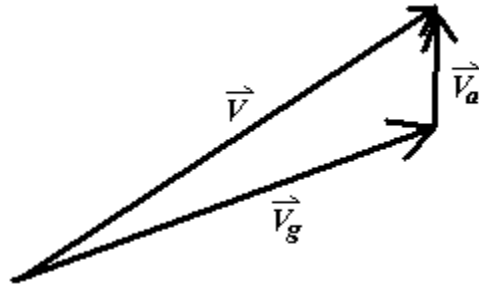
$$u_g = -\frac{\partial \psi}{\partial y} \quad (20)$$

$$v_g = \frac{\partial \psi}{\partial x} \quad (21)$$

THE AGEOSTROPHIC WIND

- The difference between the actual wind and the geostrophic wind is called the *ageostrophic wind*.

$$\vec{V}_a \equiv \vec{V} - \vec{V}_g \quad (22)$$



- Since the atmosphere is usually close to geostrophic balance, the ageostrophic wind is typically small in comparison to the geostrophic wind.
- **Horizontal divergence is a very important mechanism for rising and sinking motions in the atmosphere.**
 - Since the geostrophic wind is non-divergent, any divergence must be due to the ageostrophic wind.
 - Therefore, ***even though the ageostrophic wind is small, it is very important!***

SCALE ANALYSIS OF THE VERTICAL MOMENTUM EQUATION

- Scale analysis of the vertical momentum equation proceeds as follows (note that in this case δP is the vertical variation in pressure, which is ~ 1000 mb or 10^5 Pa).

$\frac{\partial w}{\partial t}$	$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y}$	$w \frac{\partial w}{\partial z}$	$-\frac{u^2 + v^2}{a}$	$-\frac{1}{\rho} \frac{\partial p}{\partial z}$	$2\Omega u \cos \phi$	$-g$
UW/L	UW/L	W^2/H	U^2/a	$\delta P/\rho H$	$2\Omega U \cos 45$	g
10^{-7}	10^{-7}	10^{-8}	10^{-5}	10	10^{-3}	10

$\nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$	$\nu \frac{\partial^2 w}{\partial z^2}$	$-\nu \frac{w}{a^2}$	$\nu \frac{v}{a^2} \tan \phi$	$-\nu \frac{2}{a} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$
$\nu W/L^2$	$\nu W/H^2$	$\nu W/a^2$	$\nu U/a^2$	$\nu U/La$
10^{-19}	10^{-15}	10^{-20}	10^{-18}	10^{-17}

- This analysis shows that the pressure gradient and gravity terms are dominant.
 - Therefore, on the synoptic scale, the atmosphere can be assumed to be in hydrostatic balance, and the vertical momentum equation simplifies to

$$\frac{\partial p}{\partial z} = -\rho g.$$

- A more rigorous analysis (see *Mesoscale Meteorological Modeling* by Pielke) shows that **the hydrostatic relation is appropriate if the vertical length scale is much smaller than the horizontal length scale, $H \ll L$.** This condition certainly applies on the synoptic scale.

EXERCISES

1. Perform a scale analysis of the horizontal momentum equations (in component form) for the whirlpool formed as your bathroom sink drains. Which terms are important in this case? Water has a density of 1000 kg/m^3 and a kinematic viscosity of $\nu = 1.8 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$. The horizontal pressure difference across the whirlpool is $\sim 10 \text{ Pa}$. (Use a reasonable estimate for the horizontal velocity based on your own experiences.)

2. What is the Rossby number for a tornado? Does the Coriolis force effect a tornado?

3. Expand the horizontal momentum equation

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla p - \hat{k} \times f \vec{V}$$

to show that in pure Cartesian-component form it is

$$\left(\frac{Du}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} - fv \right) \hat{i} + \left(\frac{Dv}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial y} + fu \right) \hat{j} = 0,$$

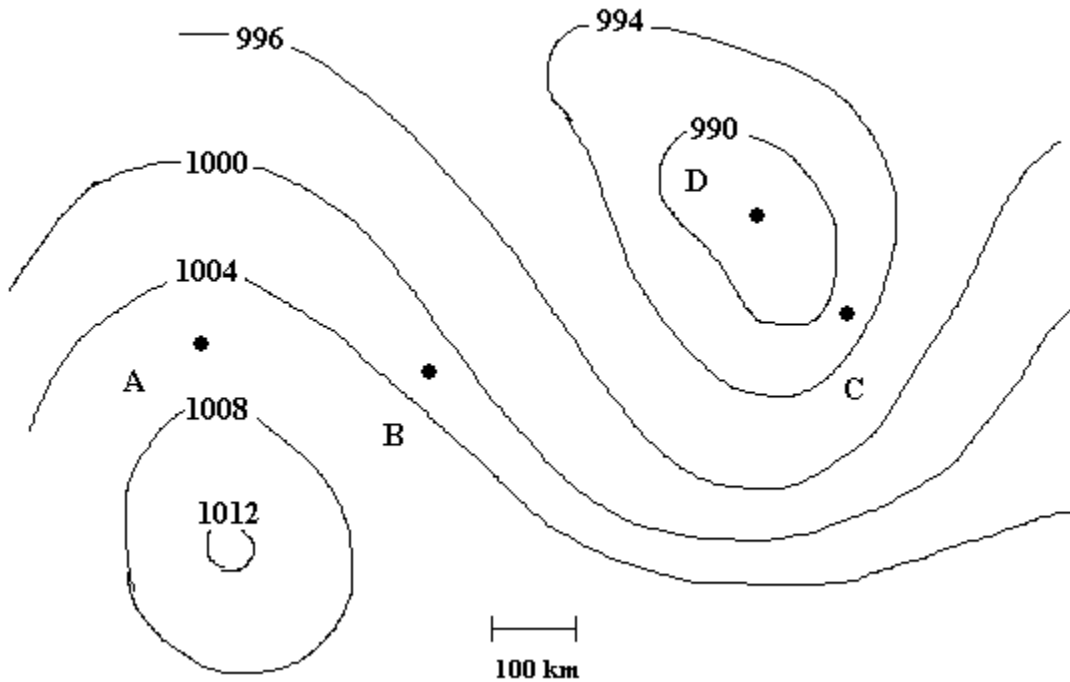
and therefore yields the two component equations

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu$$

4. Show that $\hat{k} \times (\hat{k} \times \vec{V}_g) = -\vec{V}_g$.

5. At the four points shown in the picture below, estimate the magnitude of the geostrophic wind. Assume a density of 1.23 kg/m^3 and a latitude of 45° . The isobars are labeled in mb.



6. Perform a scale analysis of the vertical momentum equation for a midlatitude thunderstorm to find out what terms can be ignored.