ESCI 342 – Atmospheric Dynamics I
Lesson 5 – The Total Derivative

THE TOTAL DERIVATIVE

- Meteorological variables such as p, T, $\vec{V}$ etc. can vary both in time and space. They are therefore functions of four independent variables, x, y, z and t.
- The differential of any of these variables (e.g., T) has the form

$$dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz .$$

  - Dividing through by the differential of time gives

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} .$$

  - By definition

$$\frac{dx}{dt} \equiv u; \quad \frac{dy}{dt} \equiv v; \quad \frac{dz}{dt} \equiv w$$

    so that

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

    which can also be written as

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{V} \bullet \nabla T .$$

- This shows that in general, ***the partial derivative is not equal to the full derivative***.
- We refer to the full derivative with respect to time as the ***total derivative*** or ***material derivative***, and give it the special notation of D/Dt, so that the total derivative operator is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V} \bullet \nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} . \quad \textbf{(1)}$$

  - In the example using temperature we therefore have

$$\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \vec{V} \bullet \nabla T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} . \quad \textbf{(2)}$$

  - ***The total derivative (D/Dt) represents the change relative to*** **a reference frame attached to the air parcel and moving with it.**
    - This is referred to as a *Lagrangian derivative*.
- ***The term ∂/∂t represents the change from a coordinate system fixed x, y, and z coordinates***. This is called the ***local derivative***, or the *Eulerian derivative.*
- The term $\vec{V} \bullet \nabla$ is called the ***advection operator***,[1] and represents that part of the local change that is due to *advection* (transport of a property due to the mass movement of the fluid).
- In meteorology we usually measure the local derivative, since our instruments are usually fixed in space. Therefore, using temperature as an example, we write

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[1] Do not confuse the advection operator, $\vec{V} \bullet \nabla$, with divergence $\nabla \bullet \vec{V}$!

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \vec{V} \bullet \nabla T \,.$$

- It is important to understand that the change we measure with our instruments may be due to either a change within the fluid itself (represented by the *DT/Dt* term), or due the movement of fluid with a different property over our instrument, represented by the $-\vec{V} \bullet \nabla T$ term.
    - o Example: The temperature at our station has been decreasing. This may be due the entire air mass losing heat due to radiation or conduction (*DT/Dt*) or due to the wind blowing colder air into our area, $-\vec{V} \bullet \nabla T$.

ADVECTION OF A SCALAR VERSUS A VECTOR

- Vector quantities can also be advected. The advection of a vector field looks like $-\vec{V} \bullet \nabla \vec{A}$.
- At first this may look odd, because we are used to the concept of the gradient operator operating on a scalar, not on a vector. But, $\nabla \vec{A}$ is indeed a defined and valid operation (contrary to what some textbooks may state), and is in fact a second-order tensor. The dot product of a vector with a second-order tensor results in a vector, so the result of $\vec{V} \bullet \nabla \vec{A}$ is just another vector.
- Since we may not want to involve ourselves with the concept of tensors, the advection of a vector is often written as $-\left(\vec{V} \bullet \nabla\right)\vec{A}$. In this form, the operator $\vec{V} \bullet \nabla$ is a scalar operator having the form

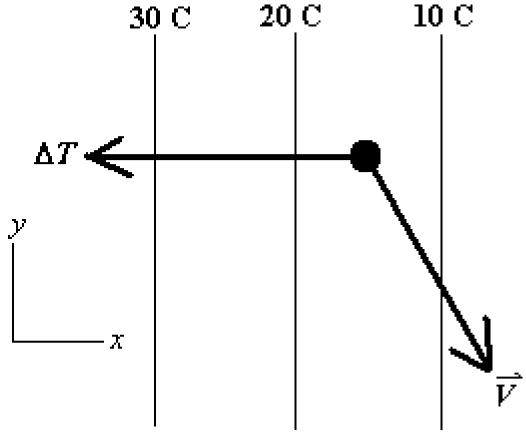
$$\vec{V} \bullet \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \,.$$

- Keep in mind that whether or not we write the advection operator with parentheses, the two forms are equivalent,

$$\left(\vec{V} \bullet \nabla\right)\vec{A} = \vec{V} \bullet \nabla\vec{A} \,. \tag{3}$$

MORE ON ADVECTION

- The advection term for a scalar involves the dot product of the velocity vector and the gradient vector. It is therefore readily evaluated.
    - o Example: The wind is from 330° at 25 m/s. The isotherms are oriented north-south as shown in the picture below, and are 100 km apart.



In component form the two vectors are

$$\vec{V} = (12.5\,m/s)\,\hat{\imath} - (21.6\,m/s)\,\hat{\jmath}$$

$$\nabla T = (-0.0001\,°C/m)\,\hat{\imath}$$

The advection is therefore

$$-\vec{V} \bullet \nabla T = -\left[(12.5\,m/s)\,\hat{\imath} - (21.6\,m/s)\,\hat{\jmath}\right] \bullet (-0.0001\,°C/m)\,\hat{\imath} = 0.00125\,°C/s\,.$$

The advection would cause the temperature at a fixed point to increase by 4.5°C in one hour, independent of any other temperature increase or decrease due to radiation or conduction.

- o Another way to solve this problem would be to find the angle between the two vectors (in this case is it 120°) and use the formula that

$$-\vec{V} \bullet \nabla T = -V|\nabla T|\cos\theta = -(25\,m/s)(0.0001°C/m)\cos 120° = 0.00125\,°C/s$$

- Advection itself is defined as $-\vec{V} \bullet \nabla s$ where s is any scalar property (e.g., u, v, w, T)
  - o The minus sign ensures that if the velocity and the gradient are opposite, then the advection is positive, since the property would be increasing with time.

ORIGIN OF THE CURVATURE TERMS

- In Lesson 3 we introduced the vector-form of the momentum equation, stating that it is frame-invariant.

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g} + \upsilon\nabla^2\vec{V}\,. \qquad \textbf{(4)}$$

- It is valid no matter what coordinate system (Cartesian, spherical, cylindrical) is chosen. It is only when we write the vector equation in component form that the choice of coordinate system becomes relevant.
- Expanding the total derivative yields

$$\frac{\partial\vec{V}}{\partial t} + \vec{V}\bullet\nabla\vec{V} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g} + \upsilon\nabla^2\vec{V}\,. \qquad \textbf{(5)}$$

- There are two terms in (5) that involve spatial derivatives of the velocity vector. These are the $\vec{V}\bullet\nabla\vec{V}$ and $\nabla^2\vec{V}$ terms. It is these terms that give rise to curvature terms, because when expanded out into coordinates, spatial derivative of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ appear.

- In Cartesian coordinates the spatial derivatives of the unit vectors are zero, because both the direction and magnitude of the unit vectors is constant in space and time.
- In other coordinate systems the unit vectors have different directions depending on the location, so their spatial derivatives are not constant.
- The origin of the curvature terms is illustrated with he term $\vec{V} \bullet \nabla \vec{V}$ from the momentum equation. This term represents the advection of momentum by the wind itself. Expanded out it has the form

$$\vec{V} \bullet \nabla \vec{V} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}. \qquad \textbf{(6)}$$

The derivatives on the right-hand-side of (6) expand as follows

$$\frac{\partial \vec{V}}{\partial x} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k} + u\frac{\partial \hat{i}}{\partial x} + v\frac{\partial \hat{j}}{\partial x} + w\frac{\partial \hat{k}}{\partial x}$$

$$\frac{\partial \vec{V}}{\partial y} = \frac{\partial u}{\partial y}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial w}{\partial y}\hat{k} + u\frac{\partial \hat{i}}{\partial y} + v\frac{\partial \hat{j}}{\partial y} + w\frac{\partial \hat{k}}{\partial y}$$

$$\frac{\partial \vec{V}}{\partial z} = \frac{\partial u}{\partial z}\hat{i} + \frac{\partial v}{\partial z}\hat{j} + \frac{\partial w}{\partial z}\hat{k} + u\frac{\partial \hat{i}}{\partial z} + v\frac{\partial \hat{j}}{\partial z} + w\frac{\partial \hat{k}}{\partial z}$$

In Cartesian coordinates the terms involving derivatives of the unit vectors would all be zero. However, in spherical coordinates the directions of the unit vectors change with position. We therefore need to evaluate all of the following derivatives:

$$\frac{\partial \hat{i}}{\partial x};\ \frac{\partial \hat{j}}{\partial x};\ \frac{\partial \hat{k}}{\partial x};\ \frac{\partial \hat{i}}{\partial y};\ \frac{\partial \hat{j}}{\partial y};\ \frac{\partial \hat{k}}{\partial y};\ \frac{\partial \hat{i}}{\partial z};\ \frac{\partial \hat{j}}{\partial z};\ \frac{\partial \hat{k}}{\partial z}.$$

This is tedious, but not difficult, with the following results:

$\frac{\partial \hat{i}}{\partial x} = \frac{\tan\phi}{a}\hat{j} - \frac{1}{a}\hat{k}$	$\frac{\partial \hat{j}}{\partial x} = -\frac{\tan\phi}{a}\hat{i}$	$\frac{\partial \hat{k}}{\partial x} = \frac{1}{a}\hat{i}$
$\frac{\partial \hat{i}}{\partial y} = 0$	$\frac{\partial \hat{j}}{\partial y} = -\frac{1}{a}\hat{k}$	$\frac{\partial \hat{k}}{\partial y} = \frac{1}{a}\hat{j}$
$\frac{\partial \hat{i}}{\partial z} = 0$	$\frac{\partial \hat{j}}{\partial z} = 0$	$\frac{\partial \hat{k}}{\partial z} = 0$

so that we have

$$\frac{\partial \vec{V}}{\partial x} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k} + u\left(\frac{\tan\phi}{a}\hat{j} - \frac{1}{a}\hat{k}\right) + -v\frac{\tan\phi}{a}\hat{i} + \frac{w}{a}\hat{i}$$

$$\frac{\partial \vec{V}}{\partial y} = \frac{\partial u}{\partial y}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial w}{\partial y}\hat{k} - \frac{v}{a}\hat{k} + \frac{w}{a}\hat{j}$$

$$\frac{\partial \vec{V}}{\partial z} = \frac{\partial u}{\partial z}\hat{i} + \frac{\partial v}{\partial z}\hat{j} + \frac{\partial w}{\partial z}\hat{k}$$

and putting these into equation (6) gives the advection of momentum in component form,

$$\vec{V}\bullet\nabla\vec{V} = \left(\vec{V}\bullet\nabla u - \frac{uv\tan\phi}{a} + \frac{uw}{a}\right)\hat{i} + \left(\vec{V}\bullet\nabla v + \frac{u^2\tan\phi}{a} + \frac{vw}{a}\right)\hat{j} + \left(\vec{V}\bullet\nabla w - \frac{\left(u^2+v^2\right)}{a}\right)\hat{k}. \quad \textbf{(7)}$$

- Through a similar, though more tedious analysis, we can show that in parameterized spherical coordinates the viscous term expands as

$$\begin{aligned}\upsilon\nabla^2\vec{V} = \upsilon&\left[\nabla^2 u - \frac{u}{a^2}\left(\tan^2\phi + 1\right) - \frac{2}{a}\frac{\partial v}{\partial x}\tan\phi + \frac{2}{a}\frac{\partial w}{\partial x}\right]\hat{i} \\ &+ \upsilon\left[\nabla^2 v - \frac{v}{a^2}\left(\tan^2\phi + 1\right) + \frac{2}{a}\frac{\partial u}{\partial x}\tan\phi + \frac{w}{a^2}\tan\phi + \frac{2}{a}\frac{\partial w}{\partial y}\right]\hat{j} \\ &+ \upsilon\left[\nabla^2 w + \frac{v}{a^2}\tan\phi - \frac{2w}{a^2} - \frac{2}{a}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]\hat{k}\end{aligned} \quad \textbf{(8)}$$

- The terms in (7) and (8) that involve the radius of the Earth, *a*, are called the **curvature terms**.
- Curvature terms will appear anytime we take spatial derivatives of a vector and expand it into components in spherical coordinates.

THE FULL MOMENTUM EQUATIONS

- The full momentum equations in component form in parameterized spherical coordinates are

$$\begin{aligned}\frac{\partial u}{\partial t} + \vec{V}\bullet\nabla u - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi \\ + \upsilon\left[\nabla^2 u - \frac{u}{a^2}\left(\tan^2\phi + 1\right) - \frac{2}{a}\frac{\partial v}{\partial x}\tan\phi + \frac{2}{a}\frac{\partial w}{\partial x}\right]\end{aligned} \quad \textbf{(9)}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + \vec{V}\bullet\nabla v + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi \\ + \upsilon\left[\nabla^2 v - \frac{v}{a^2}\left(\tan^2\phi + 1\right) + \frac{2}{a}\frac{\partial u}{\partial x}\tan\phi + \frac{w}{a^2}\tan\phi + \frac{2}{a}\frac{\partial w}{\partial y}\right]\end{aligned} \quad \textbf{(10)}$$

$$\begin{aligned}\frac{\partial w}{\partial t} + \vec{V}\bullet\nabla w - \frac{\left(u^2+v^2\right)}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi - g \\ + \upsilon\left[\nabla^2 w - \frac{2w}{a^2} + \frac{v}{a^2}\tan\phi - \frac{2}{a}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]\end{aligned} \quad \textbf{(11)}$$

- The table below summarizes the various terms in the three momentum equations

local der.	**advective terms**	**advection curvature terms**	**pressure gradient terms**	**Coriolis terms**	**gravity term**	**viscous terms**	**viscous curvature terms**
$\frac{\partial u}{\partial t}$	$\vec{V}\bullet\nabla u$	$-\frac{uv\tan\phi}{a}+\frac{uw}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$2\Omega v\sin\varphi$ $-2\Omega w\cos\varphi$		$\upsilon\nabla^2 u$	$\upsilon\begin{bmatrix}\nabla^2 u \\ -\frac{u}{a^2}\left(\tan^2\phi+1\right) \\ -\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi \\ +\frac{2}{a}\frac{\partial w}{\partial x}\end{bmatrix}$
$\frac{\partial v}{\partial t}$	$\vec{V}\bullet\nabla v$	$\frac{u^2\tan\phi}{a}+\frac{vw}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$-2\Omega u\sin\phi$		$\upsilon\nabla^2 v$	$\upsilon\begin{bmatrix}\nabla^2 v \\ -\frac{v}{a^2}\left(\tan^2\phi+1\right) \\ +\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi \\ +\frac{w}{a^2}\tan\phi \\ +\frac{2}{a}\frac{\partial w}{\partial y}\end{bmatrix}$
$\frac{\partial w}{\partial t}$	$\vec{V}\bullet\nabla w$	$-\frac{\left(u^2+v^2\right)}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$2\Omega u\cos\phi$	$-g$	$\upsilon\nabla^2 w$	$\upsilon\begin{bmatrix}\nabla^2 w \\ -\frac{2w}{a^2} \\ +\frac{v}{a^2}\tan\phi \\ -\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\end{bmatrix}$

VECTOR FORM VS. COMPONENT FORM OF MOMENTUM EQUATION

- Notice that even in spherical coordinates the momentum equation has the simple form of

$$\frac{D\vec{V}}{Dt}=-\frac{1}{\rho}\nabla p-2\vec{\Omega}\times\vec{V}+\vec{g}+\upsilon\nabla^2\vec{V} \qquad \textbf{(12)}$$

or

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V} = -\frac{1}{\rho} \nabla p - 2\vec{\Omega} \times \vec{V} + \vec{g} + \upsilon \nabla^2 \vec{V} , \qquad \textbf{(13)}$$

when written in vector form.

- The curvature terms do not appear until we start writing the momentum equations in component form.
- ***<u>It is much easier to memorize the momentum equation in vector form</u>*.**
    - o If you know it's vector form, you can always then expand it into components using your knowledge of vector calculus.

EXERCISES

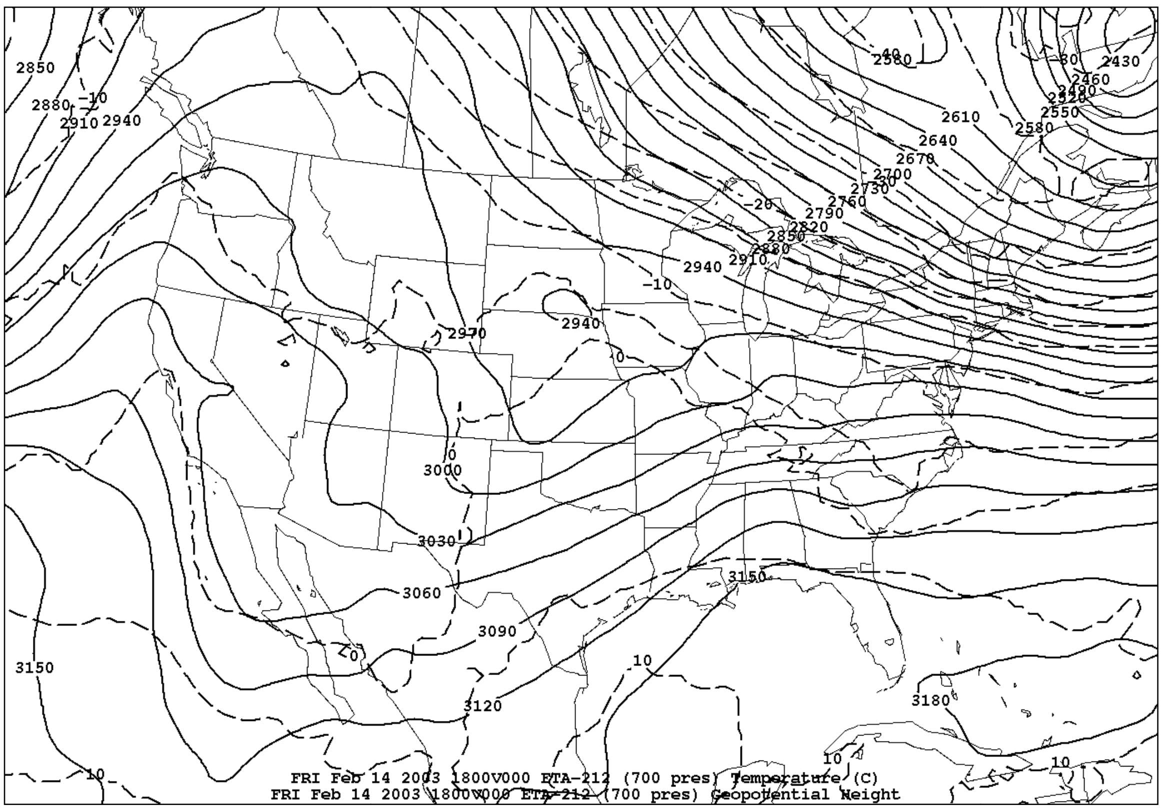
1. Show that if the wind is blowing parallel to the isotherms that the temperature advection is zero.

2. On the map below, indicate an area where:

a. The temperature would be increasing due to advection.

b. The temperature would be decreasing due to advection.

c. An area where temperature advection is very weak.



3. Show that in pure Cartesian coordinates

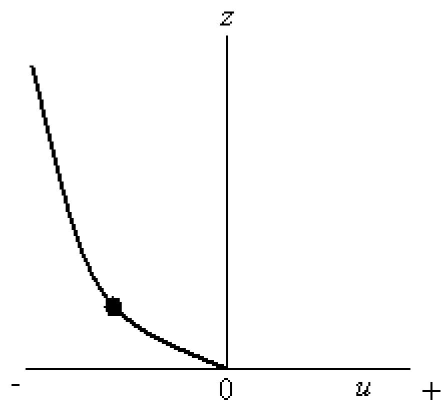
$$(\vec{V} \bullet \nabla \vec{A}) = (\vec{V} \bullet \nabla a_x)\hat{i} + (\vec{V} \bullet \nabla a_y)\hat{j} + (\vec{V} \bullet \nabla a_z)\hat{k}\,.$$

4. Show that in pure Cartesian coordinates

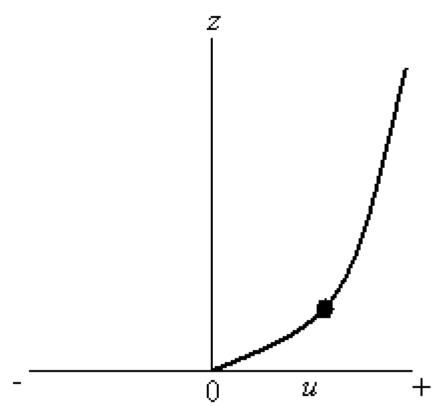
$$\frac{D\vec{V}}{Dt} = \left(\frac{\partial u}{\partial t} + \vec{V} \bullet \nabla u\right)\hat{i} + \left(\frac{\partial v}{\partial t} + \vec{V} \bullet \nabla v\right)\hat{j} + \left(\frac{\partial w}{\partial t} + \vec{V} \bullet \nabla w\right)\hat{k}$$

5. a. For the following profile of u, explain whether a downdraft would cause an increase or decrease in u at the location of the dot. Assume that u is constant in x and y [$u = u(t, z)$]

Hint: $\frac{\partial u}{\partial t} = -\vec{V} \bullet \nabla u = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z}$



b. Do the same as in 6.a., only for the following profile



6. Prove the following identities for parameterized spherical coordinates:

$\frac{\partial \hat{i}}{\partial x} = \frac{\tan\phi}{a}\hat{j} - \frac{1}{a}\hat{k}$	$\frac{\partial \hat{j}}{\partial x} = -\frac{\tan\phi}{a}\hat{i}$	$\frac{\partial \hat{k}}{\partial x} = \frac{1}{a}\hat{i}$
$\frac{\partial \hat{i}}{\partial y} = 0$	$\frac{\partial \hat{j}}{\partial y} = -\frac{1}{a}\hat{k}$	$\frac{\partial \hat{k}}{\partial y} = \frac{1}{a}\hat{j}$
$\frac{\partial \hat{i}}{\partial z} = 0$	$\frac{\partial \hat{j}}{\partial z} = 0$	$\frac{\partial \hat{k}}{\partial z} = 0$

7. Show that in parameterized spherical coordinates

$$\vec{V}\bullet\nabla\vec{V}=\left(\vec{V}\bullet\nabla u-\frac{uv\tan\phi}{a}+\frac{uw}{a}\right)\hat{i}+\left(\vec{V}\bullet\nabla v+\frac{u^2\tan\phi}{a}+\frac{vw}{a}\right)\hat{j}+\left(\vec{V}\bullet\nabla w-\frac{\left(u^2+v^2\right)}{a}\right)\hat{k}$$

8. Show that $\nabla^2\vec{V}=\nabla^2(\hat{i}u)+\nabla^2(\hat{j}v)+\nabla^2(\hat{k}w)$

9. Prove the identity $\nabla^2(ab)=a\nabla^2 b+2\nabla a\bullet\nabla b+b\nabla^2 a$

10. Derive the following expressions for parameterized spherical coordinates:

$$\nabla^2(\hat{i}u)=\hat{i}\left[\nabla^2 u-\frac{u}{a^2}\left(\tan^2\phi+1\right)\right]+\hat{j}\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi-\hat{k}\frac{2}{a}\frac{\partial u}{\partial x}$$

$$\nabla^2(\hat{j}v)=-\hat{i}\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi+\hat{j}\left[\nabla^2 v-\frac{v}{a^2}\left(\tan^2\phi+1\right)\right]+\hat{k}\left(\frac{v}{a^2}\tan\phi-\frac{2}{a}\frac{\partial v}{\partial y}\right)$$

$$\nabla^2(\hat{k}w)=\hat{i}\frac{2}{a}\frac{\partial w}{\partial x}+\hat{j}\left(\frac{w}{a^2}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial y}\right)+\hat{k}\left(\nabla^2 w-\frac{2w}{a^2}\right)$$

11. Use the results of the previous problems to show that

$$\begin{aligned}\upsilon\nabla^2\vec{V}=\upsilon&\left[\nabla^2 u-\frac{u}{a^2}\left(\tan^2\phi+1\right)-\frac{2}{a}\frac{\partial v}{\partial x}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial x}\right]\hat{i}\\&+\upsilon\left[\nabla^2 v-\frac{v}{a^2}\left(\tan^2\phi+1\right)+\frac{2}{a}\frac{\partial u}{\partial x}\tan\phi+\frac{w}{a^2}\tan\phi+\frac{2}{a}\frac{\partial w}{\partial y}\right]\hat{j}\\&+\upsilon\left[\nabla^2 w+\frac{v}{a^2}\tan\phi-\frac{w}{a^2}-\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\hat{k}\end{aligned}$$