ESCI 342 – Atmospheric Dynamics I
Lesson 4 – Pressure

GEOPOTENTIAL

- The work required to raise a unit mass from the surface of the Earth to some height z is called the *geopotential*, defined as
$$\Phi = \int_0^z g dz \,. \tag{1}$$
- The geopotential height is defined as
$$Z = \Phi / g_0 \tag{2}$$
where g_0 = 9.80665 m/s^2 and is called *standard gravity*.
- Since $g \cong g_0$, the geopotential height is approximately equal to the actual height ($z \cong Z$). However, for dynamic calculations involving the wind the geopotential height must be used for maximum accuracy, since even small deviations can lead to errors in the wind.
- Heights of pressure surfaces are reported in geopotential height rather than actual height.

THE HYDROSTATIC EQUATION

- The vertical momentum equation is[1]
$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u \cos\varphi - g + \upsilon \nabla^2 w \,. \tag{3}$$
- If the atmosphere is at rest then u, w, and dw/dt are all zero, so then this becomes
$$\frac{\partial p}{\partial z} = -\rho g \,, \tag{4}$$
which is known as the *hydrostatic equation*.
    - The hydrostatic equation is often written with a full derivative,
    $$\frac{dp}{dz} = -\rho g \,.$$
    This is allowed because
    $$\frac{dp}{dz} = \frac{\partial p}{\partial t}\frac{dt}{dz} + \frac{\partial p}{\partial x}\frac{dx}{dz} + \frac{\partial p}{\partial y}\frac{dy}{dz} + \frac{\partial p}{\partial z}$$
    and if the atmosphere is at rest then p is a function of height only so that
    $$\frac{\partial p}{\partial t} = 0; \quad \frac{\partial p}{\partial x} = 0; \quad \frac{\partial p}{\partial y} = 0$$
    and therefore
    $$\frac{\partial p}{\partial z} = \frac{dp}{dz} \,.$$

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[1] Recall we are ignoring the curvature terms in the Laplacian. See the end of Lesson 3.

Just keep in mind that there will be occasions when we will use the hydrostatic equation even when the atmosphere is not at rest, and on those occasions it is more proper to use the form with the partial derivative.

- ***The hydrostatic equation states that in an atmosphere at rest the pressure gradient force is exactly balanced by gravity*.**
- In terms of geopotential height we can write

$$\frac{dp}{dz} = \frac{dp}{dZ}\frac{dZ}{dz} = \frac{g}{g_0}\frac{dp}{dZ}$$

so that the hydrostatic equation can be written as

$$\frac{dp}{dZ} = -\rho g_0 . \qquad \textbf{(5)}$$

- The hydrostatic equation can be used to find the vertical pressure profile of an atmosphere at rest as follows:
  - o Substitute for density from the ideal gas law to get

$$\frac{1}{p}\frac{dp}{dZ} = -\frac{g_0}{R_d T} .$$

  - o Integrating vertically from the surface to some geopotential height Z we get

$$p(Z) = p_0 \exp\left[-\frac{g_0}{R_d}\int_0^z \frac{1}{T}\,dZ\right]. \qquad \textit{Pressure variation with height}$$

PRESSURE DECREASE IN AN ISOTHERMAL ATMOSPHERE

- Absolute temperature varies by only 20% or so through the troposphere, so we can get an idea how pressure changes with height by assuming a constant temperature (isothermal atmosphere). If this is done, the expression for the pressure profile becomes

$$p(Z) = p_o \exp\left(-\frac{g}{R_d T}Z\right) = p_0 \exp\left(-Z/H_P\right). \qquad \textbf{(6)}$$

- H_P is the *pressure scale height* of the atmosphere, and is a measure of how rapidly the pressure drops with height. A larger scale height means a slower rate of decrease with height.
  - o At $Z = H_p$ the pressure will have decreased to 37% of the surface value ($e^{-1} = 0.368$).
  - o The pressure scale height is the *e-folding* scale for pressure.

DENSITY PROFILE

- We can also use the hydrostatic equation and the equation of state to find how density changes with height. We first start by differentiating the ideal gas law with respect to geopotential height to get

$$\frac{dp}{dZ} = R_d\left[\rho\frac{dT}{dZ} + T\frac{d\rho}{dZ}\right].$$

From the hydrostatic equation we know that

$$\frac{dp}{dZ} = -\rho g_0 ,$$

so we can write

$$R_d \left[\rho \frac{dT}{dZ} + T \frac{d\rho}{dZ} \right] = -\rho g_0 .$$

Dividing through by $R_d T \rho$ gives

$$\frac{1}{T}\frac{dT}{dZ} + \frac{1}{\rho}\frac{d\rho}{dZ} = -\frac{g_0}{R_d T} .$$

If this is integrated from the surface to some level z we get

$$\rho(Z) = \rho_0 \frac{T_0}{T(Z)} \exp\left[-\frac{g_0}{R_d} \int_0^z \frac{1}{T} dZ \right].$$ ***Density variation with height***

- *Notice that the density and pressure profiles do not have the exact same functional dependence unless the atmosphere is isothermal* [$T(Z) = T_0$], in which case

$$\rho(Z) = \rho_0 \exp\left(-Z/H_P\right)$$
$$p(Z) = p_0 \exp\left(-Z/H_P\right).$$

THICKNESS AND THE HYPSOMETRIC EQUATION

- Equation (1) can be used to derive the *hypsometric equation* relating the average temperature in a layer to the thickness of the layer. We start with

$$\frac{1}{p}\frac{dp}{dZ} = -\frac{g_0}{R_d T}$$

which can be written as

$$1 = -\frac{R_d}{g_0}\frac{T}{p}\frac{dp}{dZ} .$$

Integrating this between two atmospheric levels

$$\int_{Z_1}^{Z_2} dZ = -\frac{R_d}{g_0}\int_{Z_1}^{Z_2} \frac{T}{p}\frac{dp}{dZ} dZ$$

gives

$$Z_2 - Z_1 = Z_\Delta = -\frac{R_d}{g_0}\int_{p_1}^{p_2} T \frac{dp}{p} ,$$

where $Z_\Delta \equiv Z_2 - Z_1$ and is called the *thickness* of the layer. Using the generalized mean-value theorem from calculus we can write this as

$$Z_\Delta = -\frac{R_d}{g_0}\bar{T}\int_{p_1}^{p_2} \frac{dp}{p}$$

where $\bar{T}$ is the layer-average temperature and is found by

$$\bar{T} = \int_{p_1}^{p_2} T \frac{dp}{p} \Bigg/ \int_{p_1}^{p_2} \frac{dp}{p} .$$

The equation for the thickness of the layer is then

$$Z_{\Delta} = -\frac{R_d}{g_0}\overline{T}\ln\frac{p_2}{p_1} = \frac{R_d}{g_0}\overline{T}\ln\frac{p_1}{p_2}$$ ***Hypsometric equation***

- The hypsometric equation tells us that the thickness between two pressure levels is directly proportional to the average temperature within the layer.
- We can use thickness as a measure of the average temperature of a layer.
- Colder layers are thinner, warmer layers are thicker.
- We can use contours of thickness in a similar manner to how we use isotherms.
- ***The hypsometric equation is how the geopotential height of a pressure surface is determined from radiosonde observations***.

THE HORIZONTAL PRESSURE GRADIENT IN PRESSURE COORDINATES

- In meteorology it is often convenient to use pressure as the vertical coordinate in place of *z* or *Z*. This requires a slightly different representation for some of the terms in the momentum equation.
- On a constant pressure surface the differential of pressure is[2]

$$dp = \left(\frac{\partial p}{\partial x}\right)_{y,z,t} dx + \left(\frac{\partial p}{\partial y}\right)_{x,z,t} dy + \left(\frac{\partial p}{\partial z}\right)_{x,y,t} dz + \left(\frac{\partial p}{\partial t}\right)_{x,y,z} dt = 0\,. \quad \textbf{(7)}$$

  - Dividing through by *dx* gives

$$\left(\frac{\partial p}{\partial x}\right)_{y,z,t} + \left(\frac{\partial p}{\partial y}\right)_{x,z,t}\frac{dy}{dx} + \left(\frac{\partial p}{\partial z}\right)_{x,y,t}\frac{dz}{dx} + \left(\frac{\partial p}{\partial t}\right)_{x,y,z}\frac{dt}{dx} = 0\,.$$

  - Since we are confined to the constant pressure surface (*p* is held constant) then we can write all the total derivatives as partial derivatives,

$$\left(\frac{\partial p}{\partial x}\right)_{y,z,t} + \left(\frac{\partial p}{\partial y}\right)_{x,z,t}\left(\frac{\partial y}{\partial x}\right)_p + \left(\frac{\partial p}{\partial z}\right)_{x,y,t}\left(\frac{\partial z}{\partial x}\right)_p + \left(\frac{\partial p}{\partial t}\right)_{x,y,z}\left(\frac{\partial t}{\partial x}\right)_p = 0$$

    which rearranges to

$$\left(\frac{\partial p}{\partial x}\right)_{y,z,t} = -\left(\frac{\partial p}{\partial y}\right)_{x,z,t}\left(\frac{\partial y}{\partial x}\right)_p - \left(\frac{\partial p}{\partial z}\right)_{x,y,t}\left(\frac{\partial z}{\partial x}\right)_p - \left(\frac{\partial p}{\partial t}\right)_{x,y,z}\left(\frac{\partial t}{\partial x}\right)_p\,. \quad \textbf{(8)}$$

- Now, if we are constrained to remain on a constant pressure surface:
  1) We can still move in the *x* direction without changing the *y* coordinate, so *x* and *y* are independent. Likewise we can move in the *y* direction without changing the *x* coordinate.) This means that $\left(\partial x/\partial y\right)_p$ and $\left(\partial y/\partial x\right)_p$ are both zero.
  2) We can remain in a fixed horizontal location *x*, and *y* even if the pressure surface itself moves around. Therefore, *x* and *t* and also *y* and *t* are independent. Therefore, $\left(\partial t/\partial x\right)_p$ and $\left(\partial t/\partial y\right)_p$ are zero.
  3) We cannot arbitrarily move in the *x* or *y* directions without changing the *z* coordinate (unless the pressure surface is level), so there is a dependence between

[2] Notice that *x*, *y*, *z*, and *t* cannot all be independent in this case, since if *dz* is nonzero, the sum cannot equal zero unless either *dx*, *dy*, or *dt* is also nonzero.

z and x, and between z and y. This means that $(\partial z/\partial x)_p \neq 0$, and also $(\partial z/\partial y)_p \neq 0$.

- Eq. (8) is therefore

$$\left(\frac{\partial p}{\partial x}\right)_{y,z} = -\left(\frac{\partial p}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_p . \qquad \textbf{(9)}$$

- Substituting from the hydrostatic equation and rearranging yields

$$\left(\frac{\partial p}{\partial x}\right)_{y,z} = \rho g\left(\frac{\partial z}{\partial x}\right)_p = \rho\left(\frac{g\partial z}{\partial x}\right)_p = \rho\left(\frac{\partial \Phi}{\partial x}\right)_p . \qquad \textbf{(10)}$$

- From (10) we see that the horizontal acceleration due to the pressure gradient force can be written in terms of geopotential on a constant pressure surface

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p ,$$

or in terms of geopotential height of a constant pressure surface

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g_0\left(\frac{\partial Z}{\partial x}\right)_p .$$

- A similar analysis can be done for the y-component of the pressure gradient force, and in vector form we have the following equivalences for the horizontal pressure gradient force.

$$-\frac{1}{\rho}\nabla_H p = -\nabla_{Hp}\Phi = -g_0\nabla_{Hp} Z \qquad \textbf{(11)}$$

where

$$\nabla_H \equiv \left(\frac{\partial}{\partial x}\right)_z \hat{i} + \left(\frac{\partial}{\partial x}\right)_z \hat{j}$$

$$\nabla_{Hp} \equiv \left(\frac{\partial}{\partial x}\right)_p \hat{i} + \left(\frac{\partial}{\partial x}\right)_p \hat{j}$$

- It is important to note the following:

1) In order for a horizontal pressure gradient to exist, a constant pressure surface must be tilted with respect to a surface of constant geopotential.

2) The greater the tilt of a constant pressure surface, the greater the horizontal pressure gradient.

THE HYDROSTATIC EQUATION IN PRESSURE COORDINATES

- To find the hydrostatic equation in pressure coordinates, we start with the hydrostatic equation in height coordinates,

$$\frac{\partial p}{\partial z} = -\rho g , \qquad \textbf{(12)}$$

divide both sides by g to get

$$\frac{\partial p}{g\partial z} = -\rho \, . \tag{13}$$

- Since we can ignore any dependence of gravity on altitude, (12) becomes

$$\frac{\partial p}{g\partial z} = \frac{\partial p}{\partial (gz)} = \frac{\partial p}{\partial \Phi} = -\rho \, . \tag{14}$$

- Inverting (14) we get

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} \, ,$$

or simply

$$\frac{\partial \Phi}{\partial p} = -\alpha \, . \tag{15}$$

- Equation (15) is the hydrostatic equation in pressure coordinates.

EXERCISES

1. Show that $\frac{dZ}{dz} = \frac{g}{g_0}$.

2. If the atmosphere was incompressible (density constant at all altitudes), 100 km thick, and had a surface pressure of 1000 mb, at what altitude would the pressure be 250 mb? Sketch the graph of pressure vs. altitude for this case and discuss how it compares with the real atmosphere.

3. If the thickness of the 1000 – 500 mb layer is 5400 m, what is the layer average temperature (in °C)?

4. Find an expression for the vertical profile of pressure in an atmosphere that has a constant lapse rate of γ. [$T(z) = T_0 - \gamma z$]

5. **a.** Use a graphing calculator or other computer program to plot your result from problem 4. Use $\gamma = 6.5$°C/km, $T_0 = 288$K, and $p_0 = 1000$ mb.

b. On the same axis plot how pressure would change in an isothermal atmosphere having $T = 288$K and $p_0 = 1000$ mb.

c. Explain from a physical perspective the difference in the plots.

6. An atmosphere has a temperature profile as a function of pressure given by $T(p) = T_0 + a\ln(p/p_0)$ where T_0 is the temperature at sea level and p_0 is the pressure at sea level

a. For this atmosphere find a general expression for the layer-average temperature for a layer lying between pressures p_1 and p_2 ($p_2 > p_1$).

b. Use the expression found in part a. to find the geopotential height of the 500 mb pressure surface (use $p_0 = 1000$ mb, $T_0 = 288$ K, and $a = 36$ K.

7. If the temperature profile is linear in height ($\gamma = -\partial T/\partial z$ = constant), find an expression for temperature as a function of pressure. Hint: Start with the chain rule,

$$\frac{dT}{dp} = \frac{dT}{dz}\frac{dz}{dp}.$$