

ESCI 342 – Atmospheric Dynamics I
Lesson 3 – Fundamental Forces II

Reading: Martin, Section 2.2

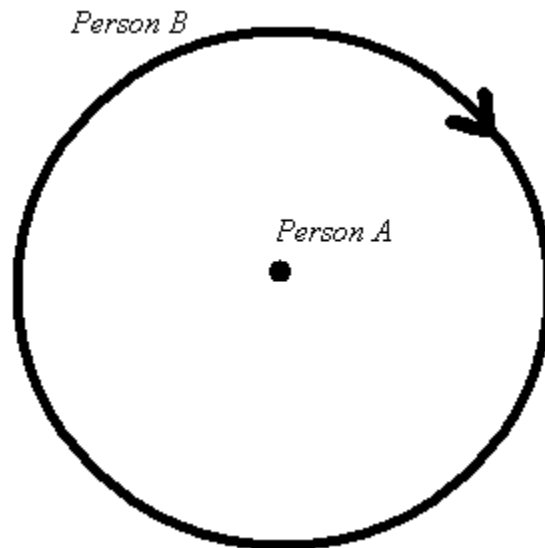
ROTATING REFERENCE FRAMES

- A reference frame in which an object with zero net force on it does not accelerate is known as an *inertial* reference frame.
- A reference frame attached to the Earth is a *noninertial* reference frame, since a net force is required to keep an object in one spot with respect to the reference frame.
- Use of a noninertial reference frame requires the introduction of *apparent* forces in order to use Newton's second law. This can be demonstrated as follows:
 - Imagine that *Person A* is on a rotating turntable while *Person B* walks in a straight line at a constant speed toward them as pictured in the left illustration below.



- Since *Person B* is moving in a straight line at constant speed, there is no acceleration and therefore no net forces acting on *Person B*.
- From the standpoint of *Person A*, who is in a rotating reference frame, *Person B* is spiraling toward him as in the illustration on the right. Therefore, *Person A* assumes that there must be a force on *Person B*, since *Person B* is accelerating. This force is an apparent force, since it only appears in the non-inertial (rotating) reference frame.

- If *Person B* is not moving in the original reference frame there is still an apparent force in the noninertial frame since *Person B* will be moving in a circle around *Person A* (from *Person A*'s reference frame) as shown below.



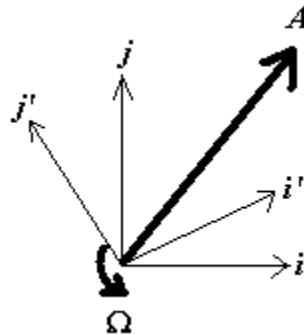
- Apparent forces are not real, though they appear to be real to someone in the noninertial reference frame.
- Apparent forces cannot do work on an object, nor can they change the speed of the object; they can only change the direction of motion of the object.

DERIVATIVES OF VECTORS IN ROTATING AND NONROTATING FRAMES

- The apparent forces can be developed mathematically as follows. Imagine two reference frames, one inertial and one rotating, sharing a common origin and a common z -axis. The rotating frame has an angular velocity of Ω (rad/s) about the z -axis.
- A vector \vec{A} can be represented in component form in either frame as

$$\vec{A} = a_x \hat{i} + a_y \hat{j} = a'_x \hat{i}' + a'_y \hat{j}' \quad (1)$$

(Primes indicate the rotating reference frame.)



- The derivative of (1) with respect to time in the nonrotating frame is

$$\frac{d\vec{A}}{dt} = \frac{da_x}{dt} \hat{i} + \frac{da_y}{dt} \hat{j} = \frac{d_a \vec{A}}{dt} \quad (2)$$

The subscript a is just a way to remind us that this is the derivative from the perspective of an observer in the *absolute* (nonrotating) reference frame.

- In the rotating frame the derivative is

$$\frac{d\vec{A}}{dt} = \frac{da'_x}{dt} \hat{i}' + \frac{da'_y}{dt} \hat{j}' + a'_x \frac{d\hat{i}'}{dt} + a'_y \frac{d\hat{j}'}{dt}. \quad (3)$$

An observer in the rotating reference frame would perceive the derivative of \vec{A} as simply

$$\frac{d_r \vec{A}}{dt} = \frac{da'_x}{dt} \hat{i}' + \frac{da'_y}{dt} \hat{j}'. \quad (4)$$

(they would be unaware of the additional terms involving the derivatives of the unit vectors). The subscript r indicates this is the derivative as perceived in the rotating frame.

Using (4) in (3) we get

$$\frac{d\vec{A}}{dt} = \frac{d_r \vec{A}}{dt} + a'_x \frac{d\hat{i}'}{dt} + a'_y \frac{d\hat{j}'}{dt}. \quad (5)$$

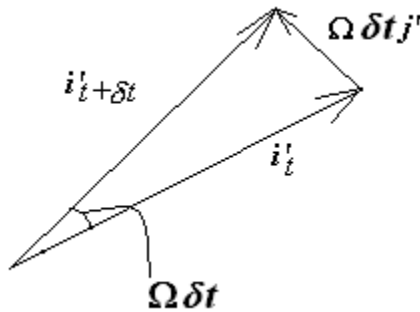
- The results from (3) and (5) must be identical, since $d\vec{A}/dt$ is a geometric invariant expression. We can therefore equate them to get

$$\frac{d_a \vec{A}}{dt} = \frac{d_r \vec{A}}{dt} + a'_x \frac{d\hat{i}'}{dt} + a'_y \frac{d\hat{j}'}{dt}. \quad (6)$$

- We need to evaluate the derivatives of the unit vectors in the rotating frame. The term $d\hat{i}'/dt$ is evaluated as follows:

$$\frac{d\hat{i}'}{dt} = \lim_{\delta t \rightarrow 0} \frac{\hat{i}'_{t+\delta t} - \hat{i}'_t}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\Omega \delta t \hat{j}'}{\delta t} = \Omega \hat{j}'$$

(refer to the diagram below).



Similarly we can show that $d\hat{j}'/dt = -\Omega \hat{i}'$. Therefore we have

$$\frac{d_a \vec{A}}{dt} = \frac{d_r \vec{A}}{dt} + a'_x \Omega \hat{j}' - a'_y \Omega \hat{i}' \quad (7)$$

which is the same as

$$\frac{d_a \vec{A}}{dt} = \frac{d_r \vec{A}}{dt} + \vec{\Omega} \times \vec{A}. \quad (8)$$

- Equation (8) shows how the derivative of a vector can be transformed between an inertial and a rotating reference frame.
 - Even though the derivation was done for a vector in only two dimensions, it works regardless of the number of dimensions.

- o Equation (8) is also valid even if the z-axes of the origins of the two coordinate systems do not coincide and/or their z-axes aren't parallel.

ACCELERATIONS IN ROTATING VERSUS NONROTATING FRAMES

- Applying (8) to the position vector of a point in space yields

$$\frac{d_a \vec{r}}{dt} = \frac{d_r \vec{r}}{dt} + \vec{\Omega} \times \vec{r}$$

which is also

$$\vec{V}_a = \vec{V}_r + \vec{\Omega} \times \vec{r} \quad (9)$$

where \vec{V}_a is the velocity in the absolute frame and \vec{V}_r is the velocity in the rotating frame.

- Applying (8) to the absolute velocity vector gives

$$\frac{d_a \vec{V}_a}{dt} = \frac{d_r \vec{V}_a}{dt} + \vec{\Omega} \times \vec{V}_a. \quad (10)$$

Substituting \vec{V}_a from (9) into the *right-hand side only* of (10) gives

$$\frac{d_a \vec{V}_a}{dt} = \frac{d_r \vec{V}_r}{dt} + \frac{d_r}{dt} (\vec{\Omega} \times \vec{r}) + \vec{\Omega} \times \vec{V}_r + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

which simplifies to

$$\frac{d_a \vec{V}_a}{dt} = \frac{d_r \vec{V}_r}{dt} + 2\vec{\Omega} \times \vec{V}_r - \Omega^2 \vec{r}. \quad (11)$$

- Equation (11) shows the relationship between the acceleration observed in the absolute frame and the acceleration observed in the rotating frame. If the two reference frames did not share a common axis then equation (11) would be

$$\frac{d_a \vec{V}_a}{dt} = \frac{d_r \vec{V}_r}{dt} + 2\vec{\Omega} \times \vec{V}_r - \Omega^2 \vec{R} \quad (12)$$

where the vector \vec{R} is a vector normal to the axis of rotation and pointing to the position of the object.

NEWTON'S SECOND LAW IN A ROTATING REFERENCE FRAME

- If we were writing Newton's second law of motion for an air parcel in the absolute reference frame we would only need to include the pressure gradient force, the gravitational force, and the viscous force, and would have

$$\frac{d_a \vec{V}_a}{dt} = -\frac{1}{\rho} \nabla p + \vec{g}^* + \nu \nabla^2 \vec{V}_a. \quad (13)$$

However, from (11) this can be written as

$$\frac{d_r \vec{V}_r}{dt} + 2\vec{\Omega} \times \vec{V}_r - \Omega^2 \vec{r} = -\frac{1}{\rho} \nabla p + \vec{g}^* + \nu \nabla^2 \vec{V}_a$$

and noting that $\nabla^2 \vec{V}_a = \nabla^2 \vec{V}_r$ (see exercises), we get

$$\frac{d_r \vec{V}_r}{dt} = -\frac{1}{\rho} \nabla p + \vec{g}^* + \nu \nabla^2 \vec{V}_r - 2\vec{\Omega} \times \vec{V}_r + \Omega^2 \vec{R} \quad (14)$$

Equation (14) is Newton's second law for the rotating coordinate system. The two additional terms that appear in the rotating coordinate system version, but not in the absolute coordinate system version, equation (13), are the accelerations due to the apparent forces.

- The first new term is the *Coriolis acceleration*, and the second is the *centrifugal acceleration*.

THE CORIOLIS FORCE

- We've seen already that the acceleration due to the Coriolis force is given by

$$\vec{a}_{cor} = -2\vec{\Omega} \times \vec{V}_r. \quad (15)$$

- Notice that the Coriolis force depends on the speed of the object relative to the rotating frame. If the object is at rest relative to the rotating frame then the Coriolis force is zero.
- Notice also that the Coriolis acts at right angles to the motion.
- Relative to the rotating reference frame at the surface of the Earth, $\vec{\Omega}$ can be written in component form as

$$\vec{\Omega} = \Omega \cos \phi \hat{j} + \Omega \sin \phi \hat{k}, \quad (16)$$

where ϕ is latitude.

The velocity vector is

$$\vec{V}_r = u\hat{i} + v\hat{j} + w\hat{k}.$$

The Coriolis acceleration therefore has components of

$$\vec{a}_{cor} = (2\Omega v \sin \phi - 2\Omega w \cos \phi)\hat{i} - 2\Omega u \sin \phi \hat{j} + 2\Omega u \cos \phi \hat{k}. \quad (17)$$

- An object moving toward the East ($u > 0$) will be deflected upward and southward due to the Coriolis force.
- An object moving toward the West ($u < 0$) will be deflected downward and northward due to the Coriolis force.
- An object moving toward the North ($v > 0$) will be deflected eastward due to the Coriolis force.
- An object moving toward the South ($v < 0$) will be deflected westward due to the Coriolis force.
- An object moving upward will be deflected westward due to the Coriolis force.
- An object moving downward will be deflected eastward due to the Coriolis force.

CENTRIFUGAL FORCE AND GRAVITY

- If an object is in circular motion in an absolute reference frame, it requires a *centripetal* acceleration of $-\Omega^2 \vec{R}$, which is directed inward toward the axis of rotation.
- In a reference frame rotating with the object, the object is not accelerating, so there is an apparent force, the *centrifugal* force, that is invoked to balance the centripetal force and keep the object from accelerating.
- The term $\Omega^2 \vec{R}$ in equation (4) represents the *centrifugal* acceleration due to the Earth's rotation.
- The centrifugal force is always directed away from the axis of rotation.

- On the Earth the centrifugal force appears to pull us away from the surface (except at the Poles) and to make us feel lighter.
 - This effect is most pronounced at the Equator.
- The centrifugal force is combined with the gravitational force to define a new force called *gravity*.
- The acceleration due to gravity force is defined as

$$\vec{g} = \vec{g}^* + \Omega^2 \vec{R}. \quad (18)$$

Equation (14) therefore becomes

$$\frac{d_r \vec{V}_r}{dt} = -\frac{1}{\rho} \nabla p - 2\vec{\Omega} \times \vec{V}_r + \vec{g} + \nu \nabla^2 \vec{V}_r. \quad (19)$$

- When you see gravity in an equation such as (19), keep in mind that it is a combination of the **gravitational** acceleration **plus centrifugal** acceleration.
- Note that gravity has a plus sign in (19), but keep in mind that if written in component form it lies solely in the negative \hat{k} direction,

$$\vec{g} = -g \hat{k}. \quad (20)$$

- Gravity is not directed exactly through the center of the Earth except at the Poles and at the Equator.
- The centrifugal force causes the Earth to not be a perfect sphere. Instead it is an *oblate spheroid*, with a larger radius at the Equator than at the Poles.

THE MOMENTUM EQUATION

- Equation (19) is known as the momentum equation.
- From now on we leave off the subscript r on the velocity, and just remember that it is the velocity relative to the Earth in our rotating coordinate system. So we will write it from now on as

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho} \nabla p - 2\vec{\Omega} \times \vec{V} + \vec{g} + \nu \nabla^2 \vec{V}. \quad (21)$$

- The terms in the momentum equation are:
 - acceleration due to the pressure gradient force: $-\frac{1}{\rho} \nabla p$
 - Coriolis acceleration: $-2\vec{\Omega} \times \vec{V}$
 - Gravity (includes centrifugal acceleration): \vec{g}
 - Friction (viscous acceleration): $\nu \nabla^2 \vec{V}$

COORDINATE REPRESENTATION OF THE MOMENTUM EQUATION

- Equation (21) is a vector equation. It is valid as is, without modification, in any coordinate system. However, for calculations and other purposes it is often necessary to write it out in component form. This is done by first choosing a coordinate system, and then expanding each term into its component form. Then, all the terms for a particular component are grouped together, resulting in three scalar equations, one for each coordinate.

- This is illustrated here using a *local Cartesian coordinate*¹ system:
- Expanding each term of (21) into this coordinate system we get:
 - For the acceleration term we have

$$\frac{d\vec{V}}{dt} = \frac{d}{dt} (u\hat{i} + v\hat{j} + w\hat{k}) = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k} + \left(u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt} \right), \quad (22)$$

but in a Cartesian frame the unit vectors are constants in both time and space, so their derivative are zero. Therefore, in Cartesian coordinates

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k}, \quad (23)$$

- The pressure-gradient term expands as

$$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{\partial p}{\partial x}\hat{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\hat{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k}. \quad (24)$$

- The Coriolis term expands as

$$-2\vec{\Omega} \times \vec{V} = (2\Omega v \sin \phi - 2\Omega w \cos \phi)\hat{i} - 2\Omega u \sin \phi \hat{j} + 2\Omega u \cos \phi \hat{k} \quad (25)$$

- The gravity term expands as

$$\vec{g} = -g\hat{k} \quad (26)$$

- The viscous term expands as

$$\nu \nabla^2 \vec{V} = \nu \nabla^2 u \hat{i} + \nu \nabla^2 v \hat{j} + \nu \nabla^2 w \hat{k} + \left(\nu u \nabla^2 \hat{i} + \nu v \nabla^2 \hat{j} + \nu w \nabla^2 \hat{k} \right), \quad (27)$$

but in Cartesian coordinates the terms in parentheses are zero, because derivatives of the unit vectors are zero. Therefore, in a Cartesian frame we have

$$\nu \nabla^2 \vec{V} = \nu \nabla^2 u \hat{i} + \nu \nabla^2 v \hat{j} + \nu \nabla^2 w \hat{k}. \quad (28)$$

- Collecting the like components (\hat{i} , \hat{j} , and \hat{k}) together we end up with the three Cartesian-component momentum equations,

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega w \cos \phi + \nu \nabla^2 u \quad (29)$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin \phi + \nu \nabla^2 v \quad (30)$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u \cos \phi - g + \nu \nabla^2 w \quad (31)$$

- In parameterized spherical coordinates the derivatives of the unit vectors, the terms in parentheses in (22) and (27), are not zero, and these lead to additional terms, called *curvature terms*, which involve the radius of the Earth (denoted by a). In parameterized spherical coordinates the component equations are

¹ Local Cartesian coordinates are coordinates based on a flat plane tangent to the Earth's surface at a particular latitude, ϕ .

$$\frac{du}{dt} - \left[\frac{uv \tan \phi}{a} - \frac{uw}{a} \right] = -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega w \cos \phi$$

$$+ \nu \left[\nabla^2 u - \frac{2}{a} \frac{\partial v}{\partial x} \tan \phi + \frac{1}{a} \frac{\partial w}{\partial x} - \frac{u}{a^2} \tan^2 \phi \right] \quad (32)$$

$$\frac{dv}{dt} + \left[\frac{u^2 \tan \phi}{a} + \frac{vw}{a} \right] = -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi$$

$$+ \nu \left[\nabla^2 v + \frac{2}{a} \frac{\partial u}{\partial x} \tan \phi - \frac{v}{a^2} (1 + \tan^2 \phi) + \frac{w}{a^2} \tan \phi + \frac{2}{a} \frac{\partial w}{\partial y} \right] \quad (33)$$

$$\frac{dw}{dt} - \left[\frac{u^2 + v^2}{a} \right] = -\frac{1}{\rho} \frac{\partial p}{\partial z} + 2\Omega u \cos \phi - g$$

$$+ \nu \left[\nabla^2 w - \frac{2}{a} \frac{\partial u}{\partial x} + \frac{v}{a^2} \tan \phi - \frac{2w}{a^2} - \frac{u}{a^2} + \frac{1}{a} \frac{\partial w}{\partial x} - \frac{2}{a} \frac{\partial v}{\partial y} \right] \quad (34)$$

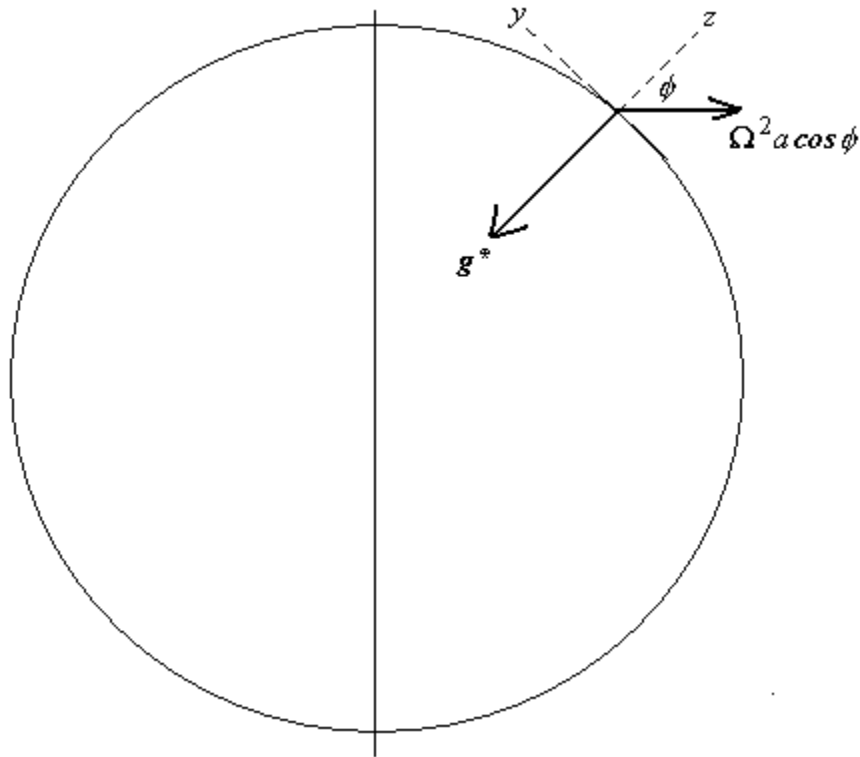
- The terms in square brackets in (32), (33), and (34) are the curvature terms, and are the only difference between the component representation in parameterized-spherical coordinates and Cartesian coordinates.

EXERCISES

1. Show that if $\vec{\Omega} = \Omega \hat{k}'$ then $a'_x \Omega \hat{j}' - a'_y \Omega \hat{i}' = \vec{\Omega} \times \vec{A}$
2. Show that if $\vec{\Omega}$ and \vec{r} are normal to each other then $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\Omega^2 \vec{r}$.
3. a. Show that the magnitude of the gravity force at latitude ϕ is given by

$$g = \sqrt{(\Omega^2 a \cos^2 \phi - g^*)^2 + (\Omega^2 a \cos \phi \sin \phi)^2}$$

where a is the radius of the Earth. (Hint: Write g^* and the centrifugal acceleration in component form and add them together. Then find the magnitude of the resultant vector. You may refer to the diagram below.)



- b. Using the results from part a., find the magnitude of the gravity force at the North Pole, 45°N , and at the Equator ($g^* = 9.81 \text{ m/s}^2$, the radius of the Earth is 6378 km , and $\Omega = 7.292 \times 10^{-5} \text{ rad/s}$).
- c. Using your results for the gravity force from part a., find the geopotential height at an altitude of 5000 meters at the North Pole, 45°N , and at the Equator (assume that the gravity force is constant with height).

4. Assume that the gravity force at the surface is $g_0 = 9.80665 \text{ m/s}^2$. Calculate the geopotential height at an altitude of 5,000 meters for the following two cases:

a. Gravity is constant with height.

b. Gravity decreases with height according to the following formula,

$$g = \frac{g_0}{(1 + z/a)^2}$$

where a is the radius of the Earth (6378 km).

c. From these results, do you think its important to include the variation of gravity with height?

5. Show that $-2\vec{\Omega} \times \vec{V} = (2\Omega v \sin \phi - 2\Omega w \cos \phi)\hat{i} - 2\Omega u \sin \phi \hat{j} + 2\Omega u \cos \phi \hat{k}$.

6. An ant is walking on a turntable that is rotating clockwise at 5 revolutions per minute (rpm). A coordinate system (x', y') is rotating with the turntable, the origin of which is the center of the turntable, with the x' and y' -axes pointing radially outward. At time $t = 0$, this coordinate system is perfectly aligned with a coordinate system fixed to the non-rotating room (x, y) . The ant is initially at coordinates $x = x' = 0$, $y = y' = 1$ cm, and with respect to the turntable is traveling along the y' -axis at a constant speed of 0.5 cm/s.

a. What is the angular velocity of the turntable in rad/s?

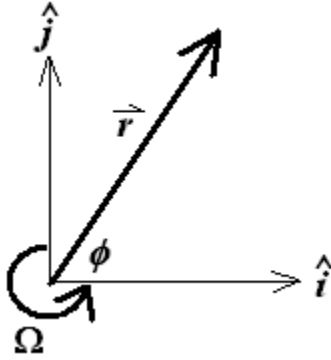
b. What are the components (in the rotating reference frame) of the ant's Coriolis acceleration at time $t = 0$?

c. What are the components (in the rotating reference frame) of the ant's centrifugal acceleration at time $t = 0$?

d. What are the components of the ant's velocity in the reference frame fixed to the room?

e. What are the components of the ant's acceleration in the reference frame fixed to the room?

7. A coordinate system is rotating counter-clockwise around its \hat{k} axis at an angular speed Ω . An object is located at position \vec{r} , and has a velocity such that the Coriolis acceleration is exactly balanced by the centrifugal acceleration.



- a. What are the u and v components of the velocity of the object in the rotating coordinate system?
 - b. What are the u and v components of the velocity of the object in a non-rotating coordinate system?
 - c. What will the path of the object look like in the rotating coordinate system?
 - d. What will the path of the object look like in a non-rotating coordinate system?
- 12.** Show that $\nabla^2 \vec{V}_a = \nabla^2 \vec{V}_r$. Hint: To do this, you will need to show that $\nabla^2 (\vec{\Omega} \times \vec{r}) = 0$. Use the identity $\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$, and recognize that $\vec{\Omega} \times \vec{r}$ is the tangential velocity of solid-body rotation. Also, the velocity divergence in solid-body rotation is zero, and the vorticity in solid-body rotation is constant.