ESCI 342 – Atmospheric Dynamics I
Lesson 2 – Fundamental Forces I

Suggested Reading: Martin, Section 2.1

UNITS

- A number is meaningless unless it is accompanied by a unit telling what the number represents.
- The standard unit system used internationally by scientists is known as the SI unit system. The basic units needed for a system of units are length, mass, and time. In the SI system, these are the meter (m), kilogram (kg), and second (s). Nearly every other unit can be derived from these three basic units. The SI unit system is sometimes referred to as the *m-k-s unit system* (as opposed to the c-g-s system, which uses centimeters, grams, and seconds as the basic units).
- Important units to remember are:

Phenomenon	Unit name	Basic units	Alternate units (non-SI)
Force	Newton (N)	kg m s^{-2}	dyne; pound
Energy	Joule (J)	N m	erg; foot-lb; calorie
Power	Watt (W)	J s^{-1}	Horsepower
Pressure	Pascal (Pa)	N m^{-2}	lb-in^{-2}; bar; torr; atmosphere; in-Hg
Temperature	Kelvin (K)	none	Celcius; Fahrenheit

- Prefixes for units:

Multiplier	Name	Abb.
10^9	giga	G
10^6	mega	M
10^3	kilo	k
10^2	hecta	h
10^1	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n

- Though internationally meteorologists adhere to SI units, in the U.S. we continue to use some traditional units that differ from SI units. Some of these are
  - Pressure: millibar (mb) = 100 Pa = hecta-Pascal (hPa)
    atmosphere (atm) = 101325 Pa = 1013.25 mb
    inches of mercury (in-Hg) – 29.92 in-Hg = 1013.25 mb = 1 atm
  - Temperature: Celcius (°C) = K – 273.15
    Fahrenheit (°F) = (9/5)°C + 32
  - Length: statute mile (mi) = 1.16 km = 1760 yds
    nautical mile (M) = 1.1 mi = 2000 yds

- Speed: Knot (kt) = nautical mile per hour = 1.14 mph ≈ 2 × m-s^{-1}
- Energy: Calorie (cal) = 4.184 J

COORDINATES AND VELOCITY

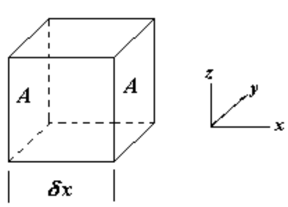
- In meteorology we use the following coordinate system:
  - The x-coordinate increases eastward
  - The y-coordinate increases northward
  - The z-coordinate increases upward
- The velocity components along each coordinate direction are defined as
  - $u \equiv dx/dt$; u is the speed in the eastward direction (*zonal* velocity)
  - $v \equiv dy/dt$; v is the speed in the northward direction (*meridional* velocity)
  - $w \equiv dz/dt$; w is the speed in the upward direction (*vertical* velocity)

FORCES

- The atmosphere obeys all the laws of physics, including Newton's second law of motion, $\vec{F} = m\vec{a}$.
- The forces acting on an air parcel are of one of two types:
  - *Body forces* – act on the center of mass, and are proportional to mass.
  - *Surface forces* – act on the surface of the parcel, and are independent of mass.
- ***In meteorology we often are using not the force, but the force per unit mass, F/m,*** which is really the acceleration.
- Meteorologists are often sloppy about whether they are talking about forces or accelerations.
  - Therefore, we often refer to the pressure gradient force when we really mean the pressure gradient acceleration.

THE PRESSURE GRADIENT FORCE

- The pressure gradient force is a surface force acting on the fluid parcel.
- For ease of derivation we often visualize the air parcel as being a cube-like shape with dimensions δx, δy, and δz.
- The pressure gradient force in the x-direction is derived as follows:



  - On the left side of the cube shown above, there will be force due to the pressure of $p_x A$ where p_x is the pressure on the left face, and A is the area.
  - On the right side of the cube the force due to the pressure is $-p_{x+\delta x} A$. Therefore, Newton's second law in the x-direction is

$$ma_{PGF\,x} = -\left(p_{x+\delta x} - p_x\right)A.$$

- If the density of the fluid is ρ then the mass is $m = \rho A \delta x$, and Newton's law is written as

$$\left(\rho A \delta x\right) a_{PGF\,x} = -\left(p_{x+\delta x} - p_x\right) A$$

  which after some rearranging becomes

$$a_{PGF\,x} = -\frac{1}{\rho}\frac{\left(p_{x+\delta x} - p_x\right)}{\delta x}.$$

- In the limit as the volume becomes infinitesimally small, we have

$$a_{PGF\,x} = -\frac{1}{\rho}\lim_{\delta x \to 0}\frac{\left(p_{x+\delta x} - p_x\right)}{\delta x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

- The derivations for the pressure gradient force in the *y*- and *z*-directions proceed the same way. Therefore, the pressure gradient force (PGF, and more appropriately called the acceleration due to the pressure gradient force) is a vector having the components

$$\vec{a}_{PGF} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\hat{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\hat{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k}.$$

- Since $\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k} = \nabla p$, the pressure gradient force can also be written as

$$\vec{a}_{PGF} = -\frac{1}{\rho}\nabla p. \qquad \textbf{(1)}$$

  - NOTE: Even though we derived (1) using Cartesian coordinates, this expression is valid in any coordinate system. It is what is called *geometric invariant*. It is only when it is expanded into components that the components will be different in different coordinate system.
- The pressure gradient force has the following properties:
  - It always is directed in the opposite direction of the pressure gradient, ∇p.
  - The stronger the pressure gradient, the stronger the pressure gradient force.
- The pressure gradient force can be estimated from maps of the isobars, as long as the distance between adjacent isobars is known, using the following approximation

$$\left|\nabla p\right| \cong \frac{\Delta p}{\Delta n}$$

  where ∇p is the contour interval for the isobars and ∇n is the horizontal distance between the isobars.

THE GRAVITATIONAL FORCE

- The gravitational force between two objects of masses *m* and *M* is given by Newton's law of gravitation,

$$\vec{F} = -\frac{GmM}{r^2}\hat{r}$$

  where *G* is the universal gravitational constant (6.673×10^{-11} N m^2 kg^{-2}), and *r* is the distance between the centers of mass of the objects ($\hat{r}$ is the unit vector along a line connecting the centers of mass of the two objects).

- The acceleration due to the gravitational force at the surface of the Earth ($r = a = 6378$ km) is

$$\vec{g}_0 * = -\frac{GM}{a^2}\hat{r}.$$

- At some altitude z above the surface of the Earth the acceleration due to the gravitational force is

$$\vec{g} * = -\frac{GM}{(a+z)^2}\hat{r}$$

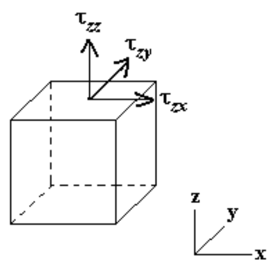
which can be written as

$$\vec{g} * = \frac{\vec{g}_0 *}{(1+z/a)^2} \cong \vec{g}_0 * \quad \text{(as long as z << a).}$$

So, **we can usually ignore changes in the gravitational acceleration with height**.

VISCOUS FORCE

- Viscous force is due to friction caused by interactions of the molecules of a fluid.
- We will present the viscous force in Cartesian coordinates.
- Imagine a cubic fluid parcel in a non-uniform flow (see diagram below). There will be three forces along each face of the cube due to the frictional effects of the fluid flow around the cube.



- The force per unit area is called the stress (denoted by τ). The notation τ_{zx} denotes the stress acting on a face at z and directed along the x-axis.
- The acceleration in the x direction on the cube due to the frictional stresses on the upper and lower faces of the cube is given by[1]

$$a_{zx} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}. \qquad \textbf{(2)}$$

- The accelerations due to the other components of the stress have a similar form. The following table summarizes the accelerations due to all the stresses.

acceleration due to stress in x-direction	$a_{xx} = \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x}$	$a_{yx} = \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y}$	$a_{zx} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}$
acceleration due to stress in y-direction	$a_{xy} = \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial x}$	$a_{yy} = \frac{1}{\rho}\frac{\partial \tau_{yy}}{\partial y}$	$a_{zy} = \frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial z}$

[1] Equation (2) is presented without derivation. Details are beyond the scope of this course. Interested students may consult an advanced fluid dynamics text.

acceleration due to stress in *z*-direction	$a_{xz} = \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial x}$	$a_{yz} = \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial y}$	$a_{zz} = \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}$

- It turns out that the stresses themselves are dependent on the flow, and are given by

stresses in *x*-direction	$\tau_{xx} = \mu\frac{\partial u}{\partial x}$	$\tau_{yx} = \mu\frac{\partial u}{\partial y}$	$\tau_{zx} = \mu\frac{\partial u}{\partial z}$
stresses in *y*-direction	$\tau_{xy} = \mu\frac{\partial v}{\partial x}$	$\tau_{yy} = \mu\frac{\partial v}{\partial y}$	$\tau_{zy} = \mu\frac{\partial v}{\partial z}$
stresses in *z*-direction	$\tau_{xz} = \mu\frac{\partial w}{\partial x}$	$\tau_{yz} = \mu\frac{\partial w}{\partial y}$	$\tau_{zz} = \mu\frac{\partial w}{\partial z}$

where μ is the *dynamic viscosity coefficient*. Therefore, the accelerations can be written as

acceleration due to stress in *x*-direction	$a_x = \upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = \upsilon\nabla^2 u$
acceleration due to stress in *y*-direction	$a_y = \upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) = \upsilon\nabla^2 v$
acceleration due to stress in *z*-direction	$a_z = \upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) = \upsilon\nabla^2 w$

where $\upsilon = \mu/\rho$ and is called the *kinematic viscosity coefficient*.

- The acceleration due to the viscous forces can be written in vector form as[2]

$$\vec{a}_{visc.} = \upsilon\nabla^2\vec{V} \qquad \textbf{(3)}$$

  - NOTE: As with the pressure gradient acceleration, even though we derived (3) using Cartesian coordinates, this expression is a *geometric invariant* and is valid in any coordinate system. However, when expanded into component form, the components in Cartesian coordinates will look very different than those in spherical coordinates.

- The viscosity of the atmosphere is small, and under most circumstances we can ignore the viscous force in our meteorological equations.

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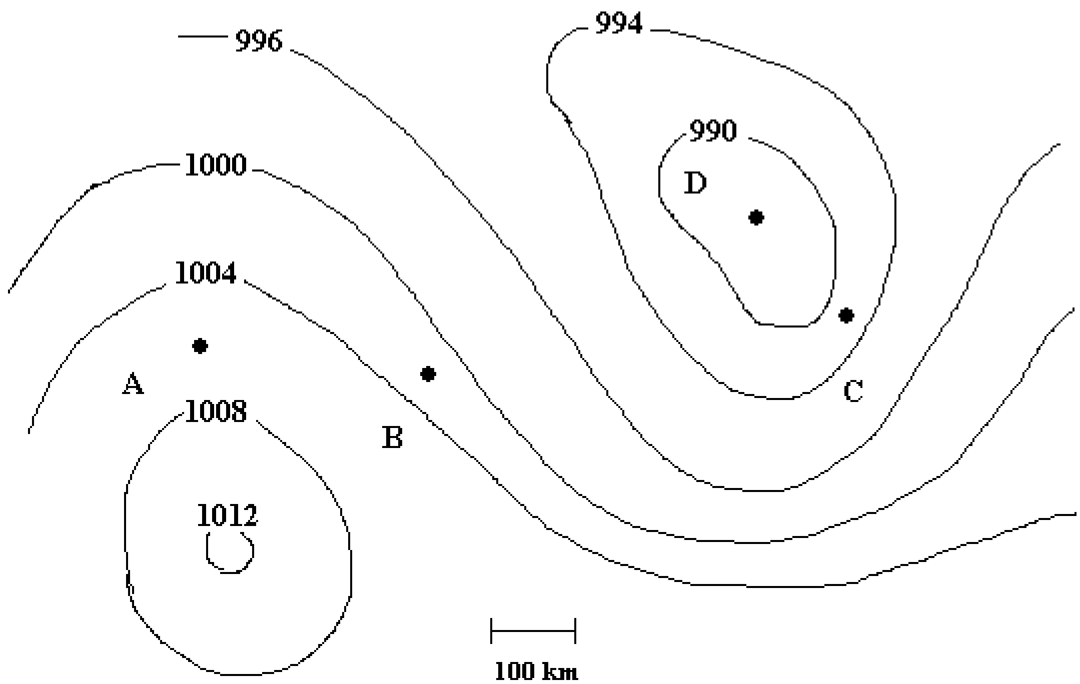
[2] A more rigorous derivation of the viscous acceleration would result in an additional term in (3) of the form $\frac{1}{3}(\upsilon + 2\upsilon')\nabla\left(\nabla\bullet\vec{V}\right)$ where a second viscous coefficient, υ', appears. Since this additional term is very small, and is inconsequential for meteorological purposes, we choose to ignore it.

EXERCISES

1. At the four points shown in the picture below, estimate the magnitude of the acceleration due to the pressure gradient force. Assume a density of 1.23 kg/m^3. The isobars are labeled in mb.



2. A man weighs 200 lb at the surface of the Earth. How much would the same man weigh at the summit of Mt. Everest (z = 8848 m)?