

## ESCI 342 – Atmospheric Dynamics I

### Lesson 1 – Vectors and Vector Calculus

**Reference:** *Schaum's Outline Series: Mathematical Handbook of Formulas and Tables*

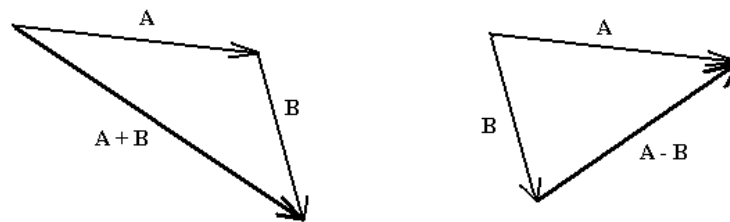
**Suggested Reading:** Martin, Section 1.2

#### COORDINATE SYSTEMS

- An *orthonormal* coordinate system is one in which the basis vectors are mutually perpendicular and are of unit length.
- *Cartesian* coordinates are an orthonormal coordinate system where the coordinate lines are not curved.
- We will be using orthonormal coordinates in this class (either Cartesian or spherical coordinates).
- The unit basis vectors for our coordinate system are  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ , which point toward the East, North, and up, respectively.

#### VECTORS

- Vectors have both a magnitude (length) and a direction.
- Vectors are denoted by either writing them in boldface ( $\mathbf{A}$ ), or by placing an arrow over the top ( $\vec{A}$ ).
- The magnitude of a vector is denoted either by  $A$  or  $|\vec{A}|$ .
- Vectors are added by placing them head to tail.



- Vector addition is
  - commutative:  $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
  - associative:  $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$
- Vectors can be multiplied by scalars. Scalar multiplication is
  - associative:  $m(n\vec{A}) = n(m\vec{A}) = mn\vec{A}$
  - distributive:  $(m+n)\vec{A} = m\vec{A} + n\vec{A}$ ;  $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$

#### COMPONENTS

- A vector can be written in terms of components along the coordinate system axes.

$$\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

- $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  are the *unit vectors* (magnitude = 1) along the  $x$ ,  $y$ , and  $z$  axes respectively.
  - The components of a vector may have negative values. For example, a velocity vector given by  $\vec{V} = -2 \text{ m s}^{-1} \hat{i} + 3 \text{ m s}^{-1} \hat{j}$  has component vectors

$$\vec{V}_x = -2 \text{ m s}^{-1} \hat{i}$$

$$\vec{V}_y = 3 \text{ m s}^{-1} \hat{j}$$

- We would think of  $\vec{V}_x$  as having a magnitude of  $2 \text{ m s}^{-1}$  and a direction opposite to  $\hat{i}$ , while  $\vec{V}_y$  has a magnitude of  $3 \text{ m s}^{-1}$  and in the direction of  $\hat{j}$ .
- Another example is gravity,  $\vec{g}$ , which in component form is

$$\vec{g} = -g \hat{k}$$

We say that the magnitude of gravity is  $g$  and its direction is opposite to  $\hat{k}$ .

- In component form, vector addition is accomplished by adding the components.

$$\vec{A} + \vec{B} = (a_x + b_x) \hat{i} + (a_y + b_y) \hat{j} + (a_z + b_z) \hat{k}$$

- In component form, multiplication by a scalar is

$$m\vec{A} = ma_x \hat{i} + ma_y \hat{j} + ma_z \hat{k}$$

- The magnitude of a vector is found from its Cartesian components (axes are normal to one another) using the Pythagorean formula

$$A = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

## DOT PRODUCT

- The dot (or scalar) product of two vectors is defined as  $\vec{A} \cdot \vec{B} = AB \cos \theta$ , where  $\theta$  is the angle between the two vectors.
- **The result of the dot product is a scalar, not a vector!**
- In component form the dot product is

$$\vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z$$

- If two vectors are normal, their dot product is zero.
- The dot product is
  - commutative:  $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
  - distributive:  $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

## CROSS PRODUCT

- The cross product is defined as  $\vec{A} \times \vec{B} = AB \sin \theta \hat{u}$ , where  $\theta$  is the angle between the two vectors and  $\hat{u}$  is the unit vector perpendicular to both  $\vec{A}$  and  $\vec{B}$  in the direction consistent with the right-hand rule.
  - Note: In European texts the cross product is often denoted using a “^” rather than a “x”.

- **The result of the cross product is a vector!**
- If two vectors are parallel, their cross product is zero.
- In orthonormal coordinates, the component form of the cross product is found by finding the determinant of a matrix whose first row is the unit vectors along the axes, and the second and third rows are the components of the vectors.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - b_y a_z) \hat{i} - (a_x b_z - b_x a_z) \hat{j} + (a_x b_y - b_x a_y) \hat{k}.$$

- **The cross product is not commutative:**  $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- The cross product is distributive:  $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

## DERIVATIVES OF VECTORS

- A vector function is a vector whose magnitude and direction depends upon other scalars (for example, time).
- The derivative of a vector function is written in component form as

$$\frac{d\vec{A}}{ds} = \frac{da_x}{ds} \hat{i} + \frac{da_y}{ds} \hat{j} + \frac{da_z}{ds} \hat{k} + a_x \frac{d\hat{i}}{ds} + a_y \frac{d\hat{j}}{ds} + a_z \frac{d\hat{k}}{ds}.$$

- The rules for differentiating dot and cross products are analogous to the product rule for scalar differentiation

$$\begin{aligned} \frac{d}{ds}(\vec{A} \cdot \vec{B}) &= \vec{A} \cdot \frac{d\vec{B}}{ds} + \frac{d\vec{A}}{ds} \cdot \vec{B} \\ \frac{d}{ds}(\vec{A} \times \vec{B}) &= \vec{A} \times \frac{d\vec{B}}{ds} + \frac{d\vec{A}}{ds} \times \vec{B} \end{aligned}$$

## THE GRADIENT

- The *del* operator in Cartesian coordinates is defined as  $\nabla \equiv \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$ .
- The del operator applied to a scalar yields a vector that points in the direction of steepest ascent (i.e., a vector that is normal to the contours and pointing toward higher values).

$$\nabla a \equiv \hat{i} \frac{\partial a}{\partial x} + \hat{j} \frac{\partial a}{\partial y} + \hat{k} \frac{\partial a}{\partial z}.$$

- When applying the del operator to a scalar field, the order of the unit vectors and the partial derivatives doesn't matter. This is why you will often see

$$\nabla a \equiv \frac{\partial a}{\partial x} \hat{i} + \frac{\partial a}{\partial y} \hat{j} + \frac{\partial a}{\partial z} \hat{k}.$$

- $\nabla a$  is called the *gradient* of  $a$ .
- **Worth repeating: The gradient is a vector that is normal to the contours and points toward higher values!**
- If the scalar is uniform in space (i.e., has the same value everywhere) then the gradient is zero.

- The del operator can also be applied to a vector (contrary to what some textbooks state), the result of which is a second-order tensor. When applying the del operator to a vector, it is important to write the unit coordinate vectors before each term rather than after it,

$$\nabla \vec{A} \equiv \hat{i} \frac{\partial \vec{A}}{\partial x} + \hat{j} \frac{\partial \vec{A}}{\partial y} + \hat{k} \frac{\partial \vec{A}}{\partial z}.$$

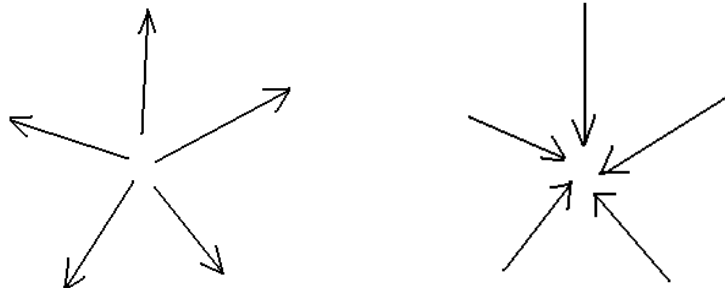
This is because the order in which vectors are directly multiplied is not commutative (i.e.,  $\hat{i} \hat{j} \neq \hat{j} \hat{i}$ ).

## DIVERGENCE

- The divergence of a vector field is defined as  $\nabla \cdot \vec{A}$ .
  - In Cartesian coordinates  $\nabla \cdot \vec{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$
- ***The divergence is a scalar!***
- When meteorologists speak of divergence, they are referring to the divergence of the velocity vector ( $\vec{V} = u \hat{i} + v \hat{j} + w \hat{k}$ ), and so we usually see divergence written as

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

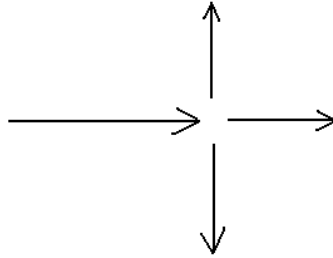
- Negative divergence is called convergence.
  - If  $\nabla \cdot \vec{V} > 0$  there is *divergence*.
  - If  $\nabla \cdot \vec{V} < 0$  there is *convergence*.
- The physical meaning of divergence can be illustrated as follows. If the vector field is pointing away from a point, the divergence at that point is positive. If the vector field is pointing into a point, the divergence at that point is negative.



- Direction alone cannot always be used to determine divergence or convergence. The vectors may be pointing in the same direction, and yet have divergence or convergence (see illustrations below).



- In many cases you cannot tell just by looking whether there is divergence or convergence. For example, the illustration below shows a case where you would have to perform the calculations to determine the divergence, since it is not obvious by examining the field.



## CURL

- The *curl* of a vector field is defined as  $\nabla \times \vec{A}$ .
- The curl is a vector whose components are found by finding the cross product of the del operator with the vector. In Cartesian coordinates, the component form the curl is

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left( \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \hat{i} - \left( \frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \right) \hat{j} + \left( \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \hat{k}.$$

- The curl of the velocity vector,  $\nabla \times \vec{V}$ , is called *vorticity*.

## THE LAPLACIAN

- The Laplacian operator is defined as  $\nabla^2 \equiv \nabla \cdot \nabla$ .
  - In Cartesian coordinates,  $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- For a scalar in Cartesian coordinates the Laplacian is  $\nabla^2 a \equiv \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}$ .
- For a vector in Cartesian coordinates the Laplacian is  $\nabla^2 \vec{A} \equiv \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$ .

## THE DEL OPERATOR IS LINEAR

- $\nabla(m+n) = \nabla m + \nabla n$
- $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$
- $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

## SPHERICAL COORDINATES

- On the Earth it makes sense to use spherical coordinates rather than Cartesian coordinates.
- In spherical coordinates for the Earth, the position of a point is given by its distance from the center of the Earth,  $r$ ; the latitude,  $\phi$ ; and the longitude,  $\lambda$ .
  - The unit vectors in spherical coordinates are the same as in Cartesian coordinates, with  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  pointing toward the East, North, and up, respectively.
- The relationships between the coordinates  $(x, y, z)$  and  $(r, \phi, \lambda)$  are

$$\begin{aligned} dx &= r \cos \phi d\lambda \\ dy &= r d\phi \\ dz &= dr \end{aligned} \tag{1}$$

- The del operator in Cartesian coordinates is

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \tag{2}$$

- The del operator expressed in spherical coordinates is derived from that for Cartesian coordinates as follows:

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} = \hat{i} \frac{\partial}{\partial \lambda} \frac{d\lambda}{dx} + \hat{j} \frac{\partial}{\partial \phi} \frac{d\phi}{dy} + \hat{k} \frac{\partial}{\partial r} \frac{dr}{dz}$$

which using the relationships in (1) gives

$$\nabla = \hat{i} \left( \frac{1}{r \cos \phi} \frac{\partial}{\partial \lambda} \right) + \hat{j} \left( \frac{1}{r} \frac{\partial}{\partial \phi} \right) + \hat{k} \frac{\partial}{\partial r}. \tag{3}$$

- The choice of which coordinate system to use (Cartesian or spherical) is strictly up to us. They both have advantages and disadvantages. Spherical coordinates are attractive because the Earth is spherical. However, the del operator is more cumbersome in spherical coordinates, and also, giving position in  $(r, \phi, \lambda)$  is more cumbersome and less intuitive than using  $(x, y, z)$ . Let's face it – Cartesian coordinates are simpler and easier to use and think about! Plus, locally the Earth appears flat to us.
- Fortunately, we can still use spherical coordinates, but express position in terms of  $x$ ,  $y$ , and  $z$  rather than  $r$ ,  $\phi$ ,  $\lambda$ . This is going to make our equations look very similar to Cartesian coordinates, although it will introduce some additional terms, called

*curvature terms*, into some of our equations. We will discuss these terms as they arise in our studies.

- o When we use spherical coordinates, but use  $x$ ,  $y$ , and  $z$  in place of  $\lambda$ ,  $\phi$ , and  $r$ , we will call this ***parameterized spherical coordinates***.

## EXERCISES

Three vectors are given by

$$\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\vec{B} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

$$\vec{C} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k}$$

1. Show that each of the following is true:

a.  $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$

b.  $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

c.  $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$

2. Show that  $\vec{A} \cdot \frac{d\vec{A}}{dt} = A \frac{dA}{dt}$  (remember that  $A$  is the magnitude of  $\vec{A}$ )

3. A vector is a function of time given by  $\vec{A}(t) = 2t \hat{i} - 3t^3 \hat{j} + 5 \ln t \hat{k}$ . Find  $\frac{d\vec{A}}{dt}$ .

4. For the following scalar fields find the magnitude of the gradient at the point indicated.

a.  $a(x, y, z) = 2x^3 y^2 - xz - z \ln y$ ;  $(x, y, z) = (4, 2, 2)$

b.  $a(x, y, z) = \cos x \sin y - z$ ;  $(x, y, z) = (0, \pi, 1)$

c.  $a(x, y) = x^2 + y^2 - 16$ ;  $(x, y) = (2, -2)$

5. For the following vector fields find the divergence at the point indicated.

a.  $\vec{A}(x, y, z) = 3xy^2 \hat{i} - xy^2 z \hat{j} - 4x^2 \ln y \hat{k}$ ;  $(x, y, z) = (4, 2, 2)$

b.  $\vec{A}(x, y, z) = \cos x \sin y \hat{i} - z \hat{j}$ ;  $(x, y, z) = (0, \pi, 1)$

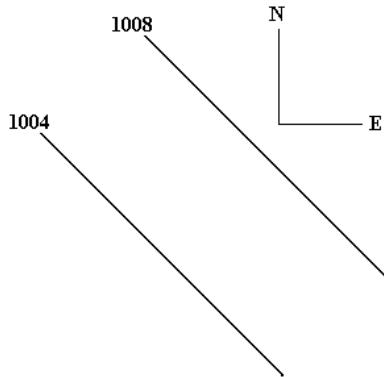
c.  $\vec{A}(x, y) = (x^2 + y^2) \hat{i} + (x^2 + y^2) \hat{j}$ ;  $(x, y) = (2, 1)$



6. The geostrophic wind is a vector field given by  $\vec{V}_g = u_g \hat{i} + v_g \hat{j}$  where  $u_g = -\frac{1}{f\rho} \frac{\partial p}{\partial y}$

and  $v_g = \frac{1}{f\rho} \frac{\partial p}{\partial x}$ .

- Show that if  $f$  and  $\rho$  are constant then the divergence of the geostrophic wind ( $\nabla \cdot \vec{V}_g$ ) is zero.
- In reality,  $f$  is not constant, but instead increases as you go north. Show that in this case the divergence of the geostrophic wind is given by  $\nabla \cdot \vec{V}_g = -v_g \frac{\partial(\ln f)}{\partial y}$ .
- For the following isobar pattern, will the geostrophic wind be convergent or divergent?



7. For the following vector fields find the curl at the point indicated.

- $\vec{A}(x, y, z) = 3xy^2 \hat{i} - xy^2 z \hat{j} - 4x^2 \ln y \hat{k}$ ;  $(x, y, z) = (4, 2, 2)$
- $\vec{A}(x, y, z) = \cos x \sin y \hat{i} - z \hat{j}$ ;  $(x, y, z) = (0, \pi, 1)$
- $\vec{A}(x, y) = (x^2 + y^2) \hat{i} + (x^2 + y^2) \hat{j}$ ;  $(x, y) = (2, 1)$

8. For the following scalar fields find the Laplacian at the point indicated.

- $a(x, y, z) = 2x^3 y^2 - xz - z \ln y$ ;  $(x, y, z) = (4, 2, 2)$
- $a(x, y, z) = \cos x \sin y - z$ ;  $(x, y, z) = (0, \pi, 1)$
- $a(x, y) = x^2 + y^2 - 16$ ;  $(x, y) = (2, -2)$

9. Show that the following identities involving the del operator are true. Do this by writing the operators and vectors in Cartesian coordinates and showing that the relationships hold. Because these identities are written in coordinate-free notation, if you prove them to be true in Cartesian coordinates, they are true in all coordinates.

$$\nabla \cdot (s\vec{A}) = \nabla s \cdot \vec{A} + s(\nabla \cdot \vec{A})$$

$$\nabla \times (s\vec{A}) = \nabla s \times \vec{A} + s(\nabla \times \vec{A})$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B})$$

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \times (\nabla \times \vec{B})$$

$$\nabla \times \nabla s = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$