ESCI 342 – Atmospheric Dynamics I
Selected Answers to Exercises for Lesson 1

2. Show that $\vec{A}\bullet\frac{d\vec{A}}{dt}=A\frac{dA}{dt}$ (remember that A is the magnitude of $\vec{A}$)

Answer:

$$\vec{A}\bullet\frac{d\vec{A}}{dt}=\left(a_x\hat{i}+a_y\hat{j}+a_z\hat{k}\right)\bullet\left(\frac{da_x}{dt}\hat{i}+\frac{da_y}{dt}\hat{j}+\frac{da_z}{dt}\hat{k}\right)=a_x\frac{da_x}{dt}+a_y\frac{da_y}{dt}+a_z\frac{da_z}{dt}$$

$$=\frac{1}{2}\frac{da_x^2}{dt}+\frac{1}{2}\frac{da_y^2}{dt}+\frac{1}{2}\frac{da_z^2}{dt}=\frac{1}{2}\frac{d}{dt}\left(a_x^2+a_y^2+a_z^2\right)=\frac{1}{2}\frac{dA^2}{dt}=A\frac{dA}{dt}$$

3. A vector is a function of time given by $\vec{A}(t)=2t\,\hat{i}-3t^3\,\hat{j}+5\ln t\,\hat{k}$. Find $\frac{d\vec{A}}{dt}$.

Answer: $\frac{d\vec{A}}{dt}=2\hat{i}-9t^2\hat{j}+\frac{5}{t}\hat{k}$

4. For the following scalar fields find the magnitude of the gradient at the point indicated.

a. $a(x,y,z)=2x^3y^2-xz-z\ln y$; $(x, y, z) = (4, 2, 2)$

$$\nabla a=\left(6x^2y^2-z\right)\hat{i}+\left(4x^3y-z/y\right)\hat{j}-\left(x+\ln y\right)\hat{k}$$

Answer: $\nabla a(4,2,2)=382\hat{i}+511\hat{j}-4.69\hat{k}$

$$\left|\nabla a(4,2,2)\right|=638$$

b. $a(x,y,z)=\cos x\sin y-z$; $(x, y, z) = (0, \pi, 1)$

$$\nabla a=-\left(\sin x\sin y\right)\hat{i}+\left(\cos x\cos y\right)\hat{j}-\hat{k}$$

Answer: $\nabla a(0,\pi,1)=0\hat{i}-\hat{j}-\hat{k}$

$$\left|\nabla a(0,\pi,1)\right|=\sqrt{2}$$

c. $a(x,y)=x^2+y^2-16$; $(x, y) = (2, -2)$

$$\nabla a=2x\,\hat{i}+2y\,\hat{j}$$

Answer: $\nabla a(2,-2)=4\hat{i}-4\hat{j}$

$$\left|\nabla a(2,-2)\right|=4\sqrt{2}$$

5. For the following vector fields find the divergence at the point indicated.

a. $\vec{A}(x, y, z) = 3xy^2\hat{i} - xy^2z\ \hat{j} - 4x^2 \ln y\ \hat{k}$; $(x, y, z) = (4, 2, 2)$

Answer: $\nabla \bullet \vec{A} = 3y^2 - 2xyz = -20$

b. $\vec{A}(x, y, z) = \cos x \sin y\ \hat{i} - z\,\hat{j}$; $(x, y, z) = (0, \pi, 1)$

Answer: $\nabla \bullet \vec{A} = -\sin x \sin y = 0$

c. $\vec{A}(x, y) = \left(x^2 + y^2\right)\hat{i} + \left(x^2 + y^2\right)\hat{j}$; $(x, y) = (2, 1)$

Answer: $\nabla \bullet \vec{A} = 2x + 2y = 6$

6. c. Answer: v_g is positive and $\partial \ln f / \partial y$ is positive, so the divergence is negative, meaning convergence.

7. For the following vector fields find the curl at the point indicated.

a. $\vec{A}(x, y, z) = 3xy^2\hat{i} - xy^2z\ \hat{j} - 4x^2 \ln y\ \hat{k}$; $(x, y, z) = (4, 2, 2)$

Answer: $\nabla \times \vec{A} = \left(xy^2 - 4x^2/y\right)\hat{i} + 8x \ln y\,\hat{j} - \left(y^2z + 6xy\right)\hat{k} = -16\hat{i} + 22.2\,\hat{j} - 56\hat{k}$

b. $\vec{A}(x, y, z) = \cos x \sin y\ \hat{i} - z\,\hat{j}$; $(x, y, z) = (0, \pi, 1)$

Answer: $\nabla \times \vec{A} = 1\hat{i} + 0\hat{j} - \cos x \cos y\,\hat{k} = 1\hat{i} + 0\,\hat{j} + 1\hat{k}$

c. $\vec{A}(x, y) = \left(x^2 + y^2\right)\hat{i} + \left(x^2 + y^2\right)\hat{j}$; $(x, y) = (2, 1)$

Answer: $\nabla \times \vec{A} = 0\hat{i} + 0\,\hat{j} + 2(x - y)\hat{k} = 0\hat{i} + 0\,\hat{j} + 2\hat{k}$

8. For the following scalar fields find the Laplacian at the point indicated.

a. $a(x, y, z) = 2x^3y^2 - xz - z \ln y$; $(x, y, z) = (4, 2, 2)$

Answer: $\nabla^2 a = 12xy^2 + 4x^3 + z/y^2 = 448.5$

b. $a(x, y, z) = \cos x \sin y - z$; $(x, y, z) = (0, \pi, 1)$

Answer: $\nabla^2 a = -2\cos x \sin y = 0$

c. $a(x, y) = x^2 + y^2 - 16$; $(x, y) = (2, -2)$

Answer: $\nabla^2 a = 4$