ESCI 341 – Atmospheric Thermodynamics
Lesson 16 – Pseudoadiabatic Processes
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SPECIFIC HEAT OF MOIST AIR

- **Water vapor is a triatomic molecule, bent at an angle of 109°. The specific heat of water vapor cannot be found by through a simple rule such as 7/2 R_v (This is because determining the energy associated with the vibrational modes is not a straightforward calculation).**
- **The heat capacities for water vapor are nearly double those for dry air, as shown below.**

	Dry Air	Water Vapor
c_v (J-kg^{-1}-K^{-1})	717	1410
c_p (J-kg^{-1}-K^{-1})	1005	1850

- **The specific heat of a mixture of ideal gases is given by**

$$c_{pm} = \frac{\sum \rho_i c_{pi}}{\sum \rho_i}.$$

- **For a mixture of dry air and water vapor this becomes**

$$c_{pm} = \frac{\rho_d c_p + \rho_v c_{pv}}{\rho_d + \rho_v} = \frac{c_p + r c_{pv}}{1 + r}.$$

- **Since r is so small, we can often neglect the contribution of the water vapor to the specific heat, and just use the specific heat of the dry air, $c_{pm} \cong c_p$ (a similar argument applies for c_{vm}).**

THERMODYNAMIC EQUATION FOR A MOIST AIR PARCEL

- **We will consider a moist air parcel that consists of three components:**
  - **Dry air of mass m_d**

- **Water vapor with mass *m_v***
- **Liquid water with mass *m_l***

- **The total entropy of the air parcel is**

$$S = S_d + S_v + S_l$$

where the subscripts *d*, *v*, and *l* refer to dry air, water vapor, and liquid water.

- **The entropy of the dry air is given by**

$$S_d = S_{d0} + m_d c_p \ln\frac{T}{T_0} - n_d R \frac{dp_d}{p_0}.$$

- **For the entropy of the water vapor, we could write a similar expression, but instead it is more convenient to realize that since entropy is a state variable, and therefore doesn't depend on the path taken between two states, we can assume that the entropy of water vapor at temperature *T* is equal to the entropy of liquid water at temperature *T* plus the latent heat of vaporization divided by *T*,**

$$S_v = S_{v0} + m_v c \ln\frac{T}{T_0} + \frac{m_v L_v}{T}$$

where *c* is the specific heat of liquid water.

- **The entropy of the liquid water is**

$$S_{l0} = S_{l0} + m_l c \ln\frac{T}{T_0}.$$

- **The total entropy of the air parcel is therefore**

$$S = S_{d0} + S_{v0} + S_{l0} + m_d c_p \ln\frac{T}{T_0} - n_d R \frac{dp_d}{p_0} + + m_v c \ln\frac{T}{T_0} + \frac{m_v L_v}{T} m_l c \ln\frac{T}{T_0},$$

which in differential form is

$$dS = \left(m_d c_p + m_v c + m_l c\right)\frac{dT}{T} - n_d R \frac{dp_d}{p_d} + d\left(\frac{m_v L_v}{T}\right).$$

- **The specific entropy of the moist air is found by dividing by the mass of moist air, *m*, to get**

$$ds = \left(\frac{m_d}{m} c_p + \frac{m_v}{m} c + \frac{m_l}{m} c\right)\frac{dT}{T} - \frac{n_d}{m} R \frac{dp_d}{p_d} + \frac{1}{m} d\left(\frac{m_v L_v}{T}\right).$$

- **Using the following identities**

$$m = m_d + m_v = m_d(1+r) = m_v\left(\frac{1+r}{r}\right)$$

$$\frac{1}{m}d\left(\frac{m_v}{T}\right) = \frac{1}{1+r}d\left(\frac{r}{T}\right)$$

we get that

$$(1+r)ds = \left[c_p + (r+r_l)c\right]\frac{dT}{T} - R_d\frac{dp_d}{p_d} + d\left(\frac{L_v r}{T}\right), \qquad \textbf{(2)}$$

where *r* is the water vapor mixing ratio (mass of water vapor per mass of dry air) and r_l is the liquid water mixing ratio (mass of liquid water per mass of dry air).

- **Equation (2) is the thermodynamic equation for moist air undergoing reversible processes.**
- **If we assume our air parcel is an isolated system, then for reversible processes *ds* = 0. So, the equation that governs an isolated moist air parcel undergoing reversible processes is**

$$\left[c_p + (r+r_l)c\right]\frac{dT}{T} - R_d\frac{dp_d}{p_d} + d\left(\frac{L_v r}{T}\right) = 0. \qquad \textbf{(3)}$$

- **For convenience we will write equation (3) as**

$$c'\frac{dT}{T} - R_d\frac{dp_d}{p_d} + d\left(\frac{L_v r}{T}\right) = 0 \qquad \textbf{(4)}$$

where

$$c' = c_p + (r+r_l)c.$$

- **Equation (4) is written in terms of the partial pressure of dry air. It would be nice to have the equation written in term of the total pressure. We can do this by expanding the differentials in equation (4) to get**

$$c'\frac{dT}{T} - R_d\frac{dp_d}{p_d} + \frac{L_v dr}{T} - \frac{L_v r}{T^2}dT = 0.$$

and using the Clausius-Clapeyron equation we can show that

$$\frac{L_v dT}{T^2} = R_v\frac{de_s}{e_s}$$

so we have

$$c'\frac{dT}{T} - R_d\frac{dp_d}{p_d} + \frac{L_v dr}{T} - rR_v\frac{de_s}{e_s} = 0\,.$$

We also know that

$$r_s = \frac{R_d}{R_v}\frac{e_s}{p_d}$$

so if we assume the parcel is saturated we can substitute this for r_s in the last term and rearrange to get

$$c'\frac{dT}{T} - R_d\frac{dp}{p_d} + \frac{L_v dr_s}{T} = 0\,.$$

We can also write

$$\frac{1}{p_d} = \frac{1}{p - e_s} = \frac{1}{p}\left(\frac{1}{1 - e_s/p}\right) \cong \frac{1}{p}\left(1 + \frac{e_s}{p}\right) = \frac{1}{p}\left(1 + \frac{R_v}{R_d}q_s\right)$$

so we end up with

$$c'\frac{dT}{T} - \left(R_d + q_s R_v\right)\frac{dp}{p} + \frac{L_v dr_s}{T} = 0\,. \qquad \textbf{(5)}$$

- **One further modification we can make to equation (5) is to multiply it through by T to get**

$$c_p\,dT - \left(R_d + q_s R_v\right)T\frac{dp}{p} + L_v dr_s = 0\,,$$

and showing that

$$\left(R_d + q_s R_v\right)T = \left(R_d + \frac{r_s R_v}{1 + r_s}\right)T = \left(\frac{R_d + r_s R_v}{1 + r_s}\right)T + \frac{r_s R_d}{1 + r_s}T = R_d T_v + \frac{r_s R_d}{1 + r_s}T \cong R_d T_v$$

so that equation (5) is approximately written as

$$c'\,dT - \alpha dp + L_v dr_s = 0\,. \qquad \textbf{(6)}$$

- **Note that equations (4) and (6) are essentially the same equation. They are both written below.**

$$c'\frac{dT}{T} - R_d\frac{dp_d}{p_d} + d\left(\frac{L_v r}{T}\right) = 0 \qquad \textbf{(4)}$$

$$c'\,dT - \alpha dp + L_v dr_s = 0\,. \qquad \textbf{(6)}$$

- **The only significant difference between the two equations is that (4) applies to any isolated, moist air parcel, whereas equation (6) assumes that the air parcel is also saturated.**

PSEUDOADIABATIC PROCESSES

- **A *pseudoadiabatic* process is an irreversible process in which a saturated air parcel is lifted, allowing the release of latent heat through condensation or deposition, but no other diabatic heating is allowed.**
- **In a pseudoadiabatic process we assume that any liquid water is immediately removed from the air parcel, so that $r_i = 0$.**
- **Pseudoadiabatic processes are important because we often approximate moist convection as a pseudoadiabatic process.**
- **There are several key parameters or features of pseudoadiabatic processes that we need to define and explore. These are:**
  - **Equivalent potential temperature**
  - **Pseudoadiabatic lapse rate**
  - **Moist static energy**

EQUIVALENT POTENTIAL TEMPERATURE

- **The equivalent potential temperature (θ_e) is defined as the temperature that the air would have if the air parcel were lifted dry adiabatically to the level of condensation, then pseudo-adiabatically to a very low pressure such that all the water vapor were condensed and removed from the parcel, and then moved adiabatically to a pressure of 1000 mb.**
- **The equivalent potential temperature can be derived from equation (4), which is**

$$c_p d\ln T - R_d d\ln p_d + d\left(\frac{L_v r}{T}\right) = 0 \qquad \textbf{(7)}$$

(we've assumed that $c' = c_p$).

- **All of the differentials in equation (7) are exact differentials, and therefore we can integrate equation (7) between two thermodynamic states and not worry about the path of the integration.**

- **For the first endpoint we choose the condensation level, which is the level when the air parcel first becomes saturated due to adiabatic lifting. At the condensation level we have**

$$T_1 = T_c$$
$$p_{d1} = p_{dc}$$
$$r_1 = r$$

where T_c is the condensation temperature, p_{dc} is the partial pressure of the dry air at the condensation level, and r is the original mixing ratio of the air parcel.

- **The other endpoint is at a pressure of 1000 mb and a mixing ratio of zero. Thus**

$$T_2 = \theta_e$$
$$p_{d2} = p_0 = 1000\,mb$$
$$r_2 = 0$$

- **Integrating equation (7) between these endpoints gives**

$$\theta_e = T_c \left(p_0 / p_{dc} \right)^{R_d/c_p} \exp\left(L_v r / c_p T_c \right). \quad \textbf{(8)}$$

- **Equivalent potential temperature is conserved in reversible, pseudoadiabatic motion.**

PSEUDOADIABATIC LAPSE RATE

- **To find the lapse rate of a rising, saturated air parcel we will start with equation (6).**
- **The derivation is easier if we write the last term in terms of saturation specific humidity q_s instead of saturation mixing ratio r_s. We could just approximate $dr_s \cong dq_s$, but to be a little more exact we will use the relation**

$$q = r/(1+r)$$

to get that

$$dr = dq/(1-q)^2 \,.$$

Thus, equation (6) becomes

$$c'\,dT - \alpha dp + L_v dq_s/(1-q_s)^2 = 0\,. \quad \textbf{(9)}$$

- **Using the fact that**

$$q_s = \varepsilon e_s / p$$

equation (9) can be written as

$$c' dT - \alpha dp + \left[L_v q_s / (1-q_s)^2\right](de_s/e_s - dp/p) = 0,$$

and using the Clausius-Clapeyron equation we get

$$\left(c' + \frac{q_s L_v^2}{(1-q_s)^2 R_v T^2}\right) dT - \left(\alpha + \frac{q_s L_v}{(1-q_s)^2 p}\right) dp = 0.$$

Dividing through by *dz* this becomes

$$\left(c' + \frac{q_s L_v^2}{(1-q_s)^2 R_v T^2}\right)\frac{dT}{dz} = \left(\alpha + \frac{q_s L_v}{(1-q_s)^2 p}\right)\frac{dp}{dz},$$

and after substituting for *dp/dz* from the hydrostatic equation, we end up with

$$\Gamma_s = -\frac{dT}{dz} = \frac{g}{c'}\left(\frac{1 + \frac{q_s L_v}{(1-q_s)^2 R_d T_v}}{1 + \frac{q_s L_v^2}{(1-q_s)^2 c' R_v T^2}}\right) \quad \textit{Pseudoadiabatic Lapse Rate} \qquad \textbf{(10)}$$

- **If we make the approximation that $c' \cong c_p$, then the pseudoadiabatic lapse rate can be written as**

$$\Gamma_s = \Gamma_d \left(\frac{1 + \frac{q_s L_v}{(1-q_s)^2 R_d T_v}}{1 + \frac{q_s L_v^2}{(1-q_s)^2 c_p R_v T^2}}\right)$$

which illustrates the following important points:

- $\Gamma_s \le \Gamma_d$
- $\Gamma_s = \Gamma_d$ **when the parcel is dry ($q_s = 0$).**

MOIST STATIC ENERGY

- **Starting with equation (6), and assuming that $c' = c_p$ we have**

$$c_p dT - \alpha dp + L_v dr_s = 0. \qquad \textbf{(6)}$$

- **Substituting *dp* from the hydrostatic equation gives**

$$c_p dT + gdz + L_v dr_s = 0,$$

or

$$dE_s = d\left(c_p T + gz + Lr_s\right) = 0.$$

- **The quantity $E_s = c_p T + gz + Lr_s$ is conserved under pseudoadiabatic motion in a hydrostatic atmosphere, and is called the *moist static energy* (compare it with the dry static energy from Lesson 7).**

EXERCISES

1. A sample of moist air at a pressure of 1000 mb and temperature of 30°C has a density of 1.13 kg/m^3.

a. What is the virtual temperature of the air sample?

b. What is the specific humidity of the air sample?

c. What is the mixing ratio of the air sample?

d. If the relative humidity is 96%, what is the saturation mixing ratio?

e. What is the vapor pressure of the air sample?

f. What is the saturation vapor pressure of the air sample?

g. What is the absolute humidity of the air sample?

h. If the air sample were heated at *constant pressure*, circle those measures of humidity that would remain constant: T_d, e, ρ_v, q, r, RH.

2. Starting with

$$c'\,dT - \alpha dp + L_v dr_s = 0$$

fill in all steps to show that the pseudoadiabatic lapse rate is

$$\Gamma_s = \frac{g}{c'}\left(\frac{1 + \dfrac{q_s L_v}{(1-q_s)^2 R_d T_v}}{1 + \dfrac{q_s L_v^2}{(1-q_s)^2 c' R_v T^2}} \right).$$