ESCI 341 – Atmospheric Thermodynamics
Lesson 8 –Stability of Dry Air

DRY ADIABATIC LAPSE RATE

- **Consider an air parcel having temperature, pressure, and specific volume of T', p', and α'.**
  - **Variables with a prime are properties of the air parcel. Variables without a prime refer to the surrounding, environmental air.**
- **The first law of thermodynamics for the air parcel is**

$$c_p dT' = dq' + \alpha' dp'$$

- **For an adiabatic process this becomes**

$$c_p dT' = \alpha' dp' \,.$$

- **We would like to know how the temperature of an air parcel would change if we lift it. What we would like to know is dT'/dz. Dividing the first law by dz, and then recognizing that we should hold all the other independent variables constant, we get for the air parcel**

$$c_p \frac{\partial T'}{\partial z} = \alpha' \frac{\partial p'}{\partial z} \,.$$

- **As the air parcel rises we assume that its pressure, p', immediately adjusts to that of its environment, p. This, along with the hydrostatic equation, gives**

$$\frac{\partial p'}{\partial z} = \frac{\partial p}{\partial z} = -\frac{g}{\alpha} \,.$$

  Therefore, we can write

$$\frac{\partial T'}{\partial z} = -\frac{\alpha'}{\alpha}\frac{g}{c_p} \,.$$

- **The density of the air parcel will be close to that of the density of the air, so $\alpha' \approx \alpha$. Therefore we have**

$$\frac{\partial T'}{\partial z} = -\frac{g}{c_p} \,.$$

- **This formula says that if you lift an air parcel adiabatically, its temperature will decrease, which makes physical sense because the parcel will be expanding.**

- **The *dry adiabatic lapse rate* is defined as**

$$\Gamma_d \equiv -\left(\frac{\partial T'}{\partial z}\right)_{adiabatic} = \frac{g}{c_p} \text{. } \textbf{\textit{Dry Adiabatic Lapse Rate}}$$

  - **Recall that lapse rate is defined with a negative sign, so that a positive lapse rate means temperature decreases with height.**
- **For dry air c_p = 1005 J kg^{-1} K^{-1}, so that Γ_d = 9.8°C/km (~10°C/km).**

BUOYANCY

- **The vertical momentum equation for an air parcel is**

$$\frac{dw'}{dt} = -\frac{1}{\rho'}\frac{\partial p}{\partial z} - g$$

- **From the hydrostatic equation we know that**

$$\frac{\partial p}{\partial z} = -\rho g$$

  so the momentum equation becomes

$$\frac{dw'}{dt} = -\frac{1}{\rho'}(-\rho g) - g = \frac{\rho - \rho'}{\rho'} g$$

  - **If the parcel is denser than the environment, the acceleration will be downward. If it is lighter than the environment, the acceleration will be upward.**
  - **Substituting for density from the ideal gas law, and assuming the pressure of the air parcel is the same as the pressure of the environment($p = p'$), we can write the acceleration in terms of temperature**

$$a_z = \frac{(T' - T)}{T} g \text{ .}$$

  - **This shows us that warm air rises and cold air sinks.**

STABILITY IN A DRY ATMOSPHERE

- **Stability refers to whether an air parcel, one moved vertically, will continue to accelerate in the direction that it was pushed (unstable), or return in the direction from which it came (stable).**

- **Imagine an air parcel that is in equilibrium with the environment, so that $T' = T = T_0$. There will be no acceleration of the air parcel.**
  - **If the air parcel is displaced, T' will change according to the adiabatic lapse rate so that $T'(z) = T_0 - \Gamma_d z$**
  - **At altitude z, the environmental temperature is $T(z) = T_0 - \gamma z$.**
  - **The acceleration at altitude z is**

$$a_z = \frac{\gamma - \Gamma_d}{T_0 - \gamma z} g z \, .$$

  - **If the acceleration is positive (upward), the parcel will continue to accelerate away from its original position, and the air is unstable.**
  - **If the acceleration is negative (downward), the parcel will begin to move downward back toward its original position, and the air is stable.**
  - **If the acceleration is zero, the parcel will remain at its new location, and the air is neutral.**
- **To assess the stability, the environmental lapse rate (γ) must be compared with the dry adiabatic lapse rate. This leads to the following stability criteria:**
    - $\Gamma_d < \gamma$ **unstable**
    - $\Gamma_d = \gamma$ **neutral**
    - $\Gamma_d > \gamma$ **stable**

LOCAL VERSUS NONLOCAL STATIC STABILITY

- **The definitions and criteria for stability given in the prior section are only valid for *local* static stability.**
- **A definition of and method for determining *nonlocal* static stability is given by Roland Stull.[1]**
- **Stull argues that it doesn't make sense to only assess static stability of an air parcel at a single level, because this doesn't account for the possibility that an air parcel's momentum may carry it through a 'stable layer' where $\Gamma_d > \gamma$, and**

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[1] **Stull, R.B., 1991: Static Stability-An Update, *Bull. Amer. Met. Soc.*, 72, 1521-1529**

into another unstable or neutral layer. Use of local stability may give an unrealistic picture of the true static stability of the air column.

- **In Stull's nonlocal approach, the entire atmospheric sounding must be evaluated.**
- **Stull's nonlocal approach is a little more complicated than the traditional, local approach, but is likely to be more relevant for many realistic conditions.**
- **The interested student is referred to Stull's paper for more details.**

POTENTIAL TEMPERATURE

- **Potential temperature (denoted as θ) is defined as the temperature an air parcel would have if it were moved dry-adiabatically to a reference pressure p_0 = 1000 mb.**
- **From Poisson's relation for T and p we get that**

$$\theta = T\left(\frac{p_0}{p}\right)^{R_d/c_p}.$$

- **If and air parcel undergoes an adiabatic process its potential temperature is conserved.**
- **The potential temperature of the environment can also be used to assess the stability. It turns out that the vertical acceleration of an air parcel can be given in terms of potential temperature,**

$$a_z = \frac{\theta' - \theta}{\theta} g.$$

- **To see this, imagine that the air parcel starts out at the same temperature (and therefore potential temperature) as its environment ($\theta = \theta' = \theta_0$). If the parcel is lifted a small distance z, then**

$$\theta(z) = \theta_0 + \frac{\partial \theta}{\partial z} z,$$

while $\theta'(z)$ continues to equal θ_0, since potential temperature is conserved in adiabatic motion. Therefore,

$$a_z = -\frac{g}{\theta}\frac{\partial \theta}{\partial z} z.$$

- **The potential temperature profile alone can be used to assess the stability of an unsaturated air parcel.**

$$\frac{\partial \theta}{\partial z} > 0; \quad stable$$
$$\frac{\partial \theta}{\partial z} = 0; \quad neutral$$
$$\frac{\partial \theta}{\partial z} < 0; \quad unstable$$

BRUNT-VAISALA FREQUENCY

- **We have previously shown that the vertical acceleration of an air parcel is**

$$a_z = -\frac{g}{\theta}\frac{\partial \theta}{\partial z} z \,.$$

- **Since** $a_z = d^2z/dt^2$ **we can write**

$$\frac{d^2z}{dt^2} = -\frac{g}{\theta}\frac{\partial \theta}{\partial z} z$$

or

$$\frac{d^2z}{dt^2} + \left(\frac{g}{\theta}\frac{\partial \theta}{\partial z}\right) z = 0 \,.$$

- **This is a 2nd order homogeneous ordinary differential equation of the form**

$$\frac{d^2z}{dt^2} + N^2 z = 0$$

where

$$N^2 = \frac{g}{\theta}\frac{\partial \theta}{\partial z} \,.$$

- **If N^2 is positive the solution is**

$$z(t) = Ae^{iNt} + Be^{-iNt}$$

and the parcel oscillates around its initial altitude at frequency *N*. In this case the atmosphere is stable.

- **If N^2 is negative the solution is**

$$z(t) = Ae^{Nt} + Be^{-Nt}$$

and the parcel accelerates away from its initial altitude. In this case the atmosphere is unstable.

- ***N* is known as the Brunt-Vaisala frequency.**
- **The Brunt-Vaisala frequency is an *angular* frequency, meaning it is actually expressed in radians per second.**
  - **The natural frequency of the parcel's oscillation is** $N/2\pi$ **.**

DRY STATIC ENERGY

- **From the first law of thermodynamics (ideal gas) for an adiabatic process we have**

$$c_p dT - \alpha dp = 0 .$$

- **Substituting *dp* from the hydrostatic equation gives**

$$c_p dT + g dz = 0 ,$$

which can be written as

$$dE = d\left(c_p T + gz\right) = 0 .$$

- **The quantity** $E = c_p T + gz$ **is conserved under adiabatic processes in a hydrostatic atmosphere. It is called the *dry static energy*, and is the sum of the specific enthalpy plus potential energy per unit mass.**
- **Dry static energy is an intensive quantity.**
- **Dry static energy can be related to potential temperature in the following manner: Start with**

$$T = \theta\left(\frac{p}{p_0}\right)^{R_d/c_p}$$

and take the differential to get

$$c_p dT = c_p \frac{T}{\theta} d\theta + R_d T \frac{dp}{p} . \qquad \textbf{(1)}$$

From the hydrostatic equation and the ideal gas law we can write

$$g dz = -\alpha dp = -R_d T \frac{dp}{p} . \qquad \textbf{(2)}$$

Adding Equations (1) and (2) we get

$$c_p dT + gdz = c_p \frac{T}{\theta} d\theta$$

or

$$dE = c_p T \, d \ln \theta \, .$$

- **A commonly used approximation relating dry static energy to potential temperature is**

  $$E \cong c_p \theta \, .$$

  This is very accurate and can be used for most, but not all, applications (for an example of when this approximation should not be used see DeCaria, A.J., 2007: Relating Static Energy to Potential Temperature: A Caution, *J. Atmos. Sci.*, **64**, 1410-1412).

EXERCISES

1. A dry air parcel has a temperature of 20°C. The environmental lapse rate is 5°C/km. The air parcel is forced to rise over a mountain that is 3 km high.

a. What is the temperature of the air parcel at the top of the mountain?

b. What is the temperature of the environment at the top of the mountain?

c. What is the buoyant acceleration of the air parcel at the top of the mountain?

d. Is the atmosphere stable or unstable?

2. For the following data, find the potential temperature at the two altitudes. Is the atmosphere stable or unstable?

Altitude (m)	Pressure (mb)	Temp (°C)	θ (K)
1480	850	7	
5700	500	-15	

3. Show that the buoyant acceleration of an air parcel can be written as

$$a_z = \frac{\theta' - \theta}{\theta} g .$$

4. Start with the ideal gas law and differentiate it with respect to *z*. Show that if the lapse rate is greater than g/R_d then density will increase with height. This lapse rate is known as the *autoconvective* lapse rate.