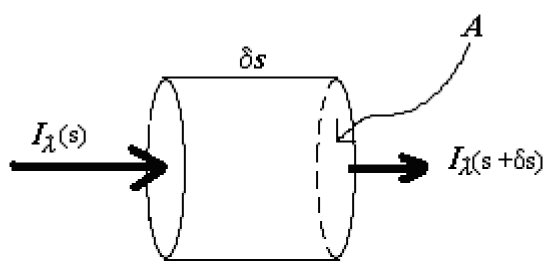
ESCI 340 – Physical Meteorology
Radiation Lesson 4 – Radiative Transfer Equation
Dr. DeCaria

References: *Atmospheric Science: An Introductory Survey*, Wallace and Hobbs
An Introduction to Atmospheric Radiation, Liou
Radiation and Cloud Processes in the Atmosphere, Liou
A First Course in Atmospheric Radiation, Petty

Reading: Petty, Chapter 7 (through Section 7.3)

EXTINCTION IN A HOMOGENOUS MEDIUM

- **If monochromatic radiation in a parallel beam passes through a homogenous medium, it will be weaker when it comes out than when it went in.**



- **The decrease in intensity is directly related to the probability that a photon of radiation will be either absorbed or scattered as it passes through the volume.**
- **If a large number of photons are passed through the volume containing a single molecule, then fraction that will be absorbed or scattered is given by**

$$P_1^{ext} = \sigma_\lambda / A, \quad \textbf{(1)}$$

where σ_λ is the molecular extinction cross section of the molecule. The subscript '1' indicates this is the probability of extinction for a single molecule.

- o **The cross section has units of m^2 or cm^2, but doesn't refer to the exact physical size of the molecule…just how big it "effectively" appears to the beam.**
- o **For instance, a small molecule that is a good absorber will remove more radiation than a large molecule that doesn't absorb very well. Therefore, from the perspective of the beam, the smaller molecule actually appears bigger, and will therefore have a bigger cross-section.**

- **The cross section has two components – a scattering cross section and an absorption cross section**

$$\sigma_\lambda = \sigma_\lambda^a + \sigma_\lambda^s. \quad \textbf{(2)}$$

- **If there are *N* molecules in the volume then the probability of extinction increases is given by**[1]

$$P_N^{ext} = 1 - \left(1 - P_1^{ext}\right)^N. \quad \textbf{(3)}$$

Using aTaylor series expansion we can write

$$\left(1 - P_1^{ext}\right)^N = 1 - NP_1^{ext} + \frac{N(N-1)}{2!}\left(P_1^{ext}\right)^2 - \frac{N(N-1)(N-2)}{3!}\left(P_1^{ext}\right)^3 + \cdots,$$

so that

$$P_N^{ext} = 1 - \left(1 - P_1^{ext}\right)^N = NP_1^{ext} - \frac{N(N-1)}{2}\left(P_1^{ext}\right)^2 + \frac{N(N-1)(N-2)}{3!}\left(P_1^{ext}\right)^3 + \cdots. \quad \textbf{(4)}$$

For very small probabilities of extinction, (4) becomes

$$P_N^{ext} \cong NP_1^{ext} = N\sigma_\lambda / A. \quad \textbf{(5)}$$

- ***N* will be given by the number density of the molecules, *n* (m^{-3}), multiplied by the volume,**

$$N = nA\delta s$$

so that we have

$$P_N^{ext} = \sigma_\lambda n \delta s. \quad \textbf{(6)}$$

- **The percentage that the intensity is reduced is negatively proportional to the probability of the photons being absorbed or scattered from the beam.**
  - **For instance, if the probability of extinction is 60%, then if we send 100 photons through the volume, we expect about 60 of them to be absorbed. Thus, the intensity of the beam would be decreased by 60%.**
- **So in general we can write**

[1] This because the probability of a photon passing through *N* molecules without being extinguished is equal to $\left(1 - P_1^{ext}\right)^N$. The probability of extinction is not linear with the number of molecules. Otherwise, the probability could exceed 100%, which is not physical. For example, if there were 3 molecules, and the probability of extinction for a single molecule were 40%, the total probability of extinction would not be 120%, but would instead be $1 - (1 - 0.4)^3 = 78.4\%$.

$$\delta I_\lambda / I_\lambda(s) = -P_N^{ext} = -\sigma_\lambda n \delta s$$

which in the limit as $\delta s \to 0$ becomes

$$dI_\lambda / I_\lambda = -\sigma_\lambda n ds \,. \qquad \textbf{(7)}$$

MASS EXTINCTION/VOLUME EXTINCTION CROSS SECTIONS

- **Sometimes the extinction cross section is given in terms of cross section per volume rather than cross section per molecule. This is achieved by multiplying the molecular extinction cross section by the number density to get a *volume extinction cross section***

$$\beta_\lambda = \sigma_\lambda n \,. \qquad \textbf{(8)}$$

  - **The volume extinction cross section will have units of area per volume, and can be though of as the total cross section of all the molecules in a unit volume.**
  - **As with the molecular extinction cross section, this will have a component due to scattering and a component due to absorption,**

$$\beta_\lambda = \beta_\lambda^a + \beta_\lambda^s \,. \qquad \textbf{(9)}$$

- **In terms of volume extinction cross section (7) becomes**

$$dI_\lambda / I_\lambda = -\beta_\lambda ds \,. \qquad \textbf{(10)}$$

- **At other times it is convenient to work with mass density instead of number density. They are related by**

$$\rho = \frac{nM}{N_A}$$

where *M* is molecular weight and N_A is Avogadro's number. So we can also write (7) or (10) as

$$dI_\lambda / I_\lambda = -k_\lambda \rho ds \,, \qquad \textbf{(11)}$$

where

$$k_\lambda = \frac{N_A}{M} \sigma_\lambda \,. \qquad \textbf{(12)}$$

- **k_λ is the *mass extinction cross-section* (or *mass extinction coefficient*) and has units of m^2/kg or cm^2/g.**

- **Like the other extinction cross section, it has a component from absorption and a component from scattering**

$$k_\lambda = k_\lambda^a + k_\lambda^s. \qquad \textbf{(13)}$$

- ***Equations* (7), (10), *and* (11) *are all COMPLETELY EQUIVALENT ways of describing the extinction through a medium*. The choice depends on personal preference, or which one makes more sense to use under the circumstances.**
- **The molecular cross section, mass cross section, and volume cross section are all related via**

$$\beta_\lambda = k_\lambda \rho = \sigma_\lambda n. \qquad \textbf{(14)}$$

EXTINCTION THROUGH A NON-HOMOGENOUS MEDIUM

- **If there are different types of gas molecules present, then each type of gas can contribute to extinction. In this case, we need to sum up the contributions from each different gas, so that Eqn. (7) becomes**

$$dI_\lambda / I_\lambda = -\left(\sum_i \sigma_{\lambda i}\, n_i \right) ds. \qquad \textbf{(15)}$$

- **By defining the total molecular extinction cross-section as a weighted sum of the individual molecular extinction cross-sections,**

$$\sigma_\lambda \equiv \frac{1}{n} \sum_i \sigma_{\lambda i}\, n_i; \quad n \equiv \sum_i n_i \qquad \textbf{(16)}$$

we can continue to use

$$dI_\lambda / I_\lambda = -\sigma_\lambda n ds \qquad \textbf{(7)}$$

for a mixture of gases.

- **If we want to use the mass extinction cross section for a mixture of gases we have**

$$k_\lambda \equiv \frac{1}{\rho} \sum_i k_{\lambda i} \rho_i \qquad \textbf{(17)}$$

where $k_{\lambda I}$ is the extinction cross section for the *i*th gas, and ρ_I is the partial density of the *i*th gas.

- **If we want to use the volume extinction cross section for a mixture of gases we have**

$$\beta_\lambda \equiv \sum_i \beta_{\lambda i} \text{ .} \qquad \textbf{(18)}$$

SOURCES OF RADIATION INTO THE BEAM

- **The radiance in the beam may also be increased by scattering into the beam, or by emission from the medium into the beam. We account for these by defining a source function coefficient, j_λ, so that the increase in radiance due to scattering and emission is given by**

$$dI_\lambda = j_\lambda \rho ds \text{ .} \qquad \textbf{(19)}$$

- **If we combine the expressions for loss and gain of radiance, we get**

$$dI_\lambda = -k_\lambda \rho I_\lambda ds + j_\lambda \rho ds \text{ .} \qquad \textbf{(20)}$$

- **If we make the following definition**

$$J_\lambda \equiv j_\lambda / k_\lambda \qquad \textbf{(21)}$$

where J_λ is the *source function* then we can write the *radiative transfer equation*

$$dI_\lambda = -k_\lambda \rho I_\lambda ds + k_\lambda \rho J_\lambda ds \text{ .} \qquad \textbf{(22)}$$

  - **J_λ has the same units as intensity (I_λ).**

BEER'S LAW

- **If the medium is non-scattering and non-emitting, then we can ignore the source function. In this case, the radiative transfer equation becomes**

$$dI_\lambda / I_\lambda = -k_\lambda \rho ds \text{ ,} \qquad \textbf{(23)}$$

(where $k_\lambda = k_\lambda^a$ since $k_\lambda^s = 0$.)

- **If we integrate this along the path from s_1 to s_2 then we get**

$$I_\lambda(s_2)/I_\lambda(s_1) = \exp\left(-\int_{s_1}^{s_2} k_\lambda \rho ds\right) \text{.} \qquad \textbf{(24)}$$

- **Defining the *path optical thickness* as**

$$\tau_{s\lambda} \equiv \int_{s_1}^{s_2} k_\lambda \rho ds, \quad \textbf{(25)}$$

we get

$$I_\lambda(s_2)/I_\lambda(s_1) = \exp(-\tau_{s\lambda}). \quad \textbf{(26)}$$

This result is known as *Beer's Law*.

- **An optical thickness of unity is where the radiation is cut to 0.3678 (which is e^{-1}) of its incident value, and can be thought of as the *e*-folding scale of the radiation.**
- **Beer's law can also be applied to the case of both scattering and absorption, assuming that scattering back into the beam can be ignored. We would just use the full mass extinction cross section, instead of just the mass absorption cross section.**

TRANSMISSIVITY, ABSORPTIVITY, AND REFLECTIVITY

- **The transmissivity (t_λ) is defined as the ratio of the transmitted radiation divided by the incident radiation, or $I_\lambda(s_2)/ I_\lambda(s_1)$. From Beer's law this is**

$$t_\lambda = \exp(-\tau_{s\lambda}). \quad \textbf{(27)}$$

- **The absorptivity (a_λ) is defined as that portion of the incident radiation that is absorbed by the medium.**
- **The reflectivity (r_λ) is defined as that portion of the incident radiation that is scattered (or reflected) directly backwards (backscattered).**
- **The following relation must hold**

$$t_\lambda + a_\lambda + r_\lambda = 1. \quad \textbf{(28)}$$

BEER'S LAW APPLIED TO A PLANE-PARALLEL ATMOSPHERE

- **If we imagine that the atmosphere is homogenous in the horizontal, but varies in the vertical, then we refer to it as a *plane-parallel atmosphere*.**
- **In the plane-parallel atmosphere, the relationship between distance along the path, *s*, and altitude, *z*, differs for upward and downward paths. These relationships are**

upward path $ds = dz/\cos\theta$

downward path $ds = -dz/\cos\theta$

- **If we define the *vertical optical thickness* between two layers as**

$$\tau_{z\lambda} \equiv \int_{z_1}^{z_2} k_\lambda \rho dz \qquad \textbf{(29)}$$

where $z_1 < z_2$, then the transmissivity is

$$t_\lambda = \exp(-\tau_{z\lambda}/\cos\theta)\,. \qquad \textbf{(30)}$$

  - **Note: If you accidentally reverse the limits of integration it is readily obvious, since you will get a negative number for optical thickness, which is not physical.**

- **If we define $\mu = |\cos\theta|$ then we have the transmissivity in a plane-parallel atmosphere between two levels as**

$$t_\lambda = \exp(-\tau_{z\lambda}/\mu) \qquad \textbf{(31)}$$

where $\tau_{z\lambda}$ is the vertical optical thickness between the two levels.

OPTICAL DEPTH

- **We will define the *vertical optical thickness* between some level *z* and the top of the atmosphere as the *optical depth* at level *z*.**
- **We will denote optical depth as $\tau_{d\lambda}$, and mathematically it is**

$$\tau_{d\lambda} = \int_{z}^{\infty} k_\lambda \rho dz\,. \qquad \textbf{(32)}$$

  - **The only difference between vertical optical thickness and optical depth is the limits of the integration [compare Eqns. (29) and (32)].**

- **Optical depth can be used as a vertical coordinate, since each level *z* will have its own value of optical depth.**
- **The optical depth is largest at the surface, and decreases to zero at the top of the atmosphere.**

- **The relationship between $d\tau_{d\lambda}$ and dz is**

$$\frac{d\tau_{d\lambda}}{dz} = -\rho k_\lambda \,, \tag{33}$$

the negative sign resulting because as z increases, the optical depth decreases.

- **The monochromatic intensity of incoming solar radiation at some level z in a plane parallel atmosphere is, from Beer's Law,**

$$I_\lambda(z)/I_\lambda(\infty) = \exp\left(-\tau_{d\lambda}/\mu\right) \tag{34}$$

where $I_\lambda(\infty)$ is the solar intensity at the top of the atmosphere, and $\tau_{d\lambda}$ is the optical depth at level z.

PATH OPTICAL THICKNESS VERSUS VERTICAL OPTICAL THICKNESS

- **We defined the *path optical thickness* as**

$$\tau_{s\lambda} \equiv \int_{s_1}^{s_2} k_\lambda \rho ds \tag{35}$$

and the *vertical optical thickness* as

$$\tau_{z\lambda} \equiv \int_{z_1}^{z_2} k_\lambda \rho dz \,. \tag{36}$$

- **Path optical thickness and vertical optical thickness are related via**

$$\tau_{s\lambda} = \tau_{z\lambda}/\mu \,. \tag{37}$$

- **Further, we defined *optical depth* as the vertical optical thickness when the upper level is the top of the atmosphere ($z_2 = \infty$).**
- **There is some inconsistency in the literature between path optical thickness, vertical optical thickness, and optical depth, so beware!**
- **For this class, to avoid confusion, we will use the subscript "*s*" when referring to path optical thickness, the subscript "*z*" when referring to vertical optical thickness, and the subscript "*d*" when referring to optical depth.**

EXERCISES

1. **You shoot a laser beam of wavelength λ horizontally through a homogeneous atmosphere of density 1.23 kg/m^3 at a target 10 km away. You find that the monochromatic radiance is half of what it started out as.**

   a. What is the path optical thickness?
   b. What is the mass extinction cross section?
   c. What is the transmissivity?
   d. What is the absorptivity (assume there is no backscattering)?

2. **The laser in problem 1 is now fired straight upward at an airplane at an altitude of 10 km. Density decreases with height according to the equation**

$$\rho(z) = \rho_0 \exp\left(-z/H\right)$$

   where *H* is 8.1 km and ρ_0 = 1.23 kg/m^3.

   a. What is the path optical thickness?
   b. What is the vertical optical thickness?
   c. What are the transmissivity and absorptivity (assume there is no backscattering)?

3. **The airplane is still at an altitude of 10 km, but has moved 5 km away horizontally.**
   a. Now what is the path optical thickness?
   b. What is the vertical optical thickness?
   c. What are the transmissivity and absorptivity (assume there is no backscattering)?

4. **In which problem (1, 2, or 3) was the path optical thickness greatest? In which problem was it least? Does an increase in path optical thickness necessarily mean an increase in actual distance?**

5. A mixture of gas A and gas B has a total number density of 1×10^{19} cm^{-3}. The particulars of the mixture are

	Gas A	**Gas B**
% by volume	**75**	**25**
M (g/mol)	**20.0**	**30.0**
σ_λ^a **(cm^2)**	**10^{-36}**	**10^{-22}**
σ_λ^s **(cm^2)**	**10^{-28}**	**10^{-28}**

a. What is the total molecular extinction cross section of the mixture?

b. What is the total mass extinction cross section of the mixture?

c. What is the total volume extinction cross section of the mixture?

d. What physical distance within the gas corresponds to a path optical thickness of 1? (You may need to find the density of the mixture using the ideal gas law. The specific gas constant for the mixture is the universal gas constant [8.3145 J-mol^{-1}-K^{-1}] divided by the molecular weight of the mixture.