ESCI 340 - Cloud Physics and Precipitation Processes
Lesson 3 - Stability and Buoyancy
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References:

Glossary of Meteorology, 2nd ed., American Meteorological Society
A Short Course in Cloud Physics, 3rd ed., Rogers and Yau, Ch. 3

Parcel Theory

- An ***air parcel*** is a conceptual model of an identifiable mass of air.
- Air parcels may be imagined to be of any shape. Depending on the context we often think of air parcels as being spherical, cylindrical, or cubical.
- Properties of the air parcel are denoted with a prime ($'$) symbol, while properties of the surrounding environment do not have primes. For example, T', p', and ρ' are the temperature, pressure, and density of the air parcel, while T, p, and ρ are the temperature, pressure, and density of the surrounding environment.

  CAUTION!! Rogers and Yau use the prime symbol to denote properties of the environment, while I and most other sources use the prime symbol for properties of the air parcel. When comparing my equations with those of the textbook you should keep this difference in mind.
- We often imagine that the air parcel is a closed system, meaning that there is no mixing between the parcel and the environment. The mass of the air parcel is assumed to remain constant. The parcel can exchange heat with its surroundings, but not mass.
- Parcel theory is merely an approximation to what actually happens in the atmosphere. However, it is a useful conceptual model, and for short time scales can be a good approximation to real atmospheric processes.

Vertical Force Balance

- The net acceleration of an air parcel is the sum of the accelerations from the individual forces acting on the parcel,

$$\vec{a} = \sum_i \vec{a}_i$$

- Clouds form from vertical motion of air parcels, so it is the vertical component of acceleration that we are most interested in. The net vertical acceleration is

$$a_z = \sum_i a_{z_i} \tag{1}$$

- The vertical accelerations acting on an air parcel are:

  Vertical pressure gradient acceleration: This is the acceleration caused by the difference in atmospheric pressure on the bottom versus the top of the air parcel.

  - The atmospheric pressure on the bottom of the parcel is greater than that on the top. Thus, the vertical pressure gradient acceleration is always directed upward.
  - The vertical pressure gradient acceleration is given by the formula

$$a_{PG} = -\frac{1}{\rho'}\frac{\partial p}{\partial z}. \tag{2}$$

  - Note that since pressure always decreases with height then $\frac{\partial p}{\partial z}$ is always negative. Thus, the vertical pressure gradient acceleration is always positive, meaning it is always directed upward.

  Gravity: Gravity is the combination of the ***gravitational*** and ***centrifugal*** accelerations.

  - Gravity is strongest at the Poles and weakest at the Equator.
  - To account for the change in gravity with latitude we use the concept of ***geopotential height***, defined as

$$Z = \frac{g}{g_0}z,$$

    where $g_0 = 9.80665$ m s^{-2}, and is called ***standard gravity***.
  - Any equation written in terms of gravity, g, and actual height, z, can be written in terms of geopotential height by simply replacing all occurrences of z with Z, and all occurrences of g with g_0.
  - Although gravitation, and thus gravity, decreases with height, the change through the troposphere is very small. Therefore, we often treat gravity as a constant with altitude, $\frac{\partial g}{\partial z} = 0$.

- Equation (1) applied to an air parcel is therefore

$$a'_z = -\frac{1}{\rho'}\frac{\partial p}{\partial z} - g, \tag{3}$$

  or in terms of geopotential height,

$$a'_z = -\frac{1}{\rho'}\frac{\partial p}{\partial Z} - g_0, \tag{4}$$

Hydrostatic Balance

- In an atmosphere that is at rest the acceleration of the air parcel would be zero.

- Also, in an atmosphere at rest, the parcel's density would be the same as that of the environment, $\rho' = \rho$.

- Thus, for an atmosphere at rest (3) becomes the ***hydrostatic equation***,

$$\frac{\partial p}{\partial z} = -\rho g. \tag{5}$$

- Even thought the hydrostatic equation technically is only valid for an atmosphere at rest, under many real-world circumstances the two terms on the right-hand side of (3) are orders of magnitude larger than the acceleration itself.
  - Under these conditions the acceleration on the left-hand side can be ignored, and the hydrostatic equation can be used even for atmospheres not at rest. This is called the ***hydrostatic approximation***.
  - The conditions for using the hydrostatic approximation are that the horizontal length scale L of the circulation being studied must be an order of magnitude or more larger than the vertical length scale H.
  - For synoptic scale circulations this criteria is certainly met, and we often use the hydrostatic approximation.
  - For cloud-scale circulations the hydrostatic approximation is not valid.

Partial vs. Full Derivatives in the Hydrostatic Equation

In previous courses having only Calculus I for a prerequisite the hydrostatic equation was written as

$$\frac{dp}{dz} = -\rho g, \tag{6}$$

without using partial derivatives. Even in more advanced courses (6) is often used in place of (5). In this section we explain why we can often substitute the full derivative for the partial derivative.

- Pressure is a function of the four independent variables, x, y, z, and t.

- The differential of pressure is therefore

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz + \frac{\partial p}{\partial t}dt. \tag{7}$$

- Dividing (7) by dz and rearranging results in

$$\frac{dp}{dz} = \frac{\partial p}{\partial z} + \frac{\partial p}{\partial x}\frac{dx}{dz} + \frac{\partial p}{\partial y}\frac{dy}{dz} + \frac{\partial p}{\partial t}\frac{dt}{dz}. \tag{8}$$

- Equation (8) shows that in general $\frac{dp}{dz} \neq \frac{\partial p}{\partial z}$.

- However, not all of the terms in (8) are the same size. If any terms are significantly smaller than others we can safely ignore these smaller terms.
- We evaluate which terms we can ignore using a process called ***scale analysis***.
  - We first recognize that
$$\frac{dx}{dz} = \frac{dx}{dt}\frac{dt}{dz} = \frac{u}{w},$$
where u is the x-component of velocity, and w is the z-component, or vertical velocity. By the same argument,
$$\frac{dy}{dz} = \frac{dy}{dt}\frac{dt}{dz} = \frac{v}{w}.$$
  - Equation (8) can therefore be written as
$$\frac{dp}{dz} = \frac{\partial p}{\partial z} + \frac{\partial p}{\partial x}\frac{u}{w} + \frac{\partial p}{\partial y}\frac{v}{w} + \frac{\partial p}{\partial t}\frac{1}{w}. \tag{9}$$
  - In scale analysis we then pick representative values for the various terms in the equation, based on the particular size or scale of the circulation. The table below shows some typical values for the terms in (9) for various scales.

Term	Convective-cloud scale	Synoptic scale
$\frac{\partial p}{\partial z}$	$\frac{10^5 \text{ Pa}}{10^4 \text{ m}} = 10 \text{ Pa m}^{-1}$	$\frac{10^5 \text{ Pa}}{10^4 \text{ m}} = 10 \text{ Pa m}^{-1}$
$\frac{\partial p}{\partial x}\frac{u}{w}$; $\frac{\partial p}{\partial y}\frac{v}{w}$	$\frac{10^2 \text{ Pa}}{10^4 \text{ m}}\frac{10 \text{ m s}^{-1}}{10 \text{ m s}^{-1}} = 10^{-2} \text{ Pa m}^{-1}$	$\frac{10^3 \text{ Pa}}{10^6 \text{ m}}\frac{10 \text{ m s}^{-1}}{10^{-2} \text{ m s}^{-1}} = 1 \text{ Pa m}^{-1}$
$\frac{\partial p}{\partial t}\frac{1}{w}$	$\frac{10^2 \text{ Pa}}{10^3 \text{ s}}\frac{1}{10 \text{ m s}^{-1}} = 10^{-2} \text{ Pa m}^{-1}$	$\frac{10^3 \text{ Pa}}{10^5 \text{ s}}\frac{1}{10 \text{ m s}^{-2}} = 10^{-1} \text{ Pa m}^{-3}$

- The table shows that through a range of scales the $\frac{\partial p}{\partial z}$ term dominates the other terms on the right-hand side of (9). Therefore we are often justified replacing $\frac{\partial p}{\partial z}$ with $\frac{dp}{dz}$ in the hydrostatic equation.

Buoyancy

- If the surrounding environment is in hydrostatic balance, then the acceleration on the air parcel from (3) becomes
$$a'_z = -\frac{1}{\rho'}\frac{\partial p}{\partial z} - g = -\frac{1}{\rho'}(-\rho g) - g,$$
or
$$a'_z = \frac{\rho - \rho'}{\rho'}g. \tag{10}$$
  - If the parcel's density is greater than the surrounding air then $\rho - \rho' < 0$, and the acceleration is negative (downward).

- If the parcel's density is less than the surrounding air then $\rho - \rho' > 0$, and the acceleration is positive (upward).

- Substituting for density from the ideal gas law, and remembering that $p' = p$, (10) can be written in terms of temperature as[1]

$$a'_z = \frac{T' - T}{T} g. \tag{11}$$

  - If the parcel's temperature is colder than the surrounding air then $T' - T < 0$, and the acceleration is negative (downward).
  - If the parcel's temperature is warmer than the surrounding air then $T' - T > 0$, and the acceleration is positive (upward).
  - For a moist air parcel we simply substitute virtual temperature for temperature,

$$a'_z = \frac{T'_v - T_v}{T_v} g. \tag{12}$$

- The acceleration of a parcel can also be written in terms of potential temperature by using the Poisson relation for the environment and the parcel,

$$\theta = T \left(\frac{p_0}{p}\right)^{R_d/c_p} ; \ \theta' = T' \left(\frac{p_0}{p'}\right)^{R_d/c_p} .$$

  The result is

$$a'_z = \frac{\theta' - \theta}{\theta} g. \tag{13}$$

  - If the parcel's potential temperature is colder than the surrounding air then $\theta' - \theta < 0$, and the acceleration is negative (downward).
  - If the parcel's potential temperature is warmer than the surrounding air then $\theta' - \theta > 0$, and the acceleration is positive (upward).

Archimedes' Principle

- We've already seen that there are several different ways to view buoyancy and vertical acceleration:

  - As a balance between vertical pressure gradient acceleration and gravity, Eq. (3).
  - In terms of the density of the parcel vs. its surroundings, Eq. (10).
  - In terms of the temperature of the parcel vs. its surroundings, Eq. (11).
  - In terms of the potential temperature of the parcel vs. its surroundings, Eq. (13).

[1] A reminder that the primed terms in my equation (11) are different than that of Rogers and Yau's (4.11) and 4.12). They put the primes on the environmental properties, while I put them on the parcel's properties.

- These are all completely equivalent, as they were all derived from the same starting point.
- Another equivalent way to view buoyancy is to begin with (10), and multiply it by the mass, $m' = \rho' V$ of the air parcel,

$$m' a'_z = m' \frac{\rho - \rho'}{\rho'} g = \rho' V \frac{\rho - \rho'}{\rho'} g = \rho V g - \rho' V g, \tag{14}$$

  where V is the volume of the air parcel.
- Equation (14) has an interesting interpretation:
  - The left hand side is the net force, F_z on the parcel.
  - The term $\rho' g V$ is simply the weight of the air parcel.
  - The term $\rho g V$ is the weight of the air displaced by the air parcel (the weight of the air that would be there if the air parcel wasn't there).
- Thus, there is an upward force on the parcel equal to the weight of the air that is displaced.
  - This is known as ***Archimedes' Principle***, and is the reason that ships (and people) float in water, and that balloons float in air.

Static Stability of Dry Air

- Static stability refers to whether or not an air parcel that has been displaced vertically will return to its original level, or will accelerate away.
- Static stability is important for determining whether or not clouds will form, and of which type.
- The conditions for stability/instability are derived by starting assuming that the air parcel begins at altitude $z = 0$, and that the air parcel and the environment are both initially at temperature $T(0) = T'(0) = T_0$.
- If the air parcel is displaced upward by a distance z its new temperature will be given by the MacLaren series

$$T'(z) = T_0 + \frac{\partial T'}{\partial z} z + \frac{1}{2!} \frac{\partial^2 T'}{\partial z^2} z^2 + \frac{1}{3!} \frac{\partial^3 T'}{\partial z^3} z^3 + \cdots ,$$

  which when truncated after the first two terms is

$$T'(z) = T_0 - \Gamma_d z, \tag{15}$$

  where Γ_d is the dry-adiabatic lapse rate.

- The temperature of the environment at the parcels new altitude is given similarly by the truncated MacLaren series

$$T(z) = T_0 - \gamma z, \tag{16}$$

  where γ is the environmental lapse rate.

- Putting (15) and (16) into (11) yields

$$a'_z = \frac{\gamma - \Gamma_d}{T_0 - \gamma z} g z. \tag{17}$$

- From (17) we see that:
  - if $\Gamma_d < \gamma$ then the acceleration is positive, and the parcel will accelerate upward, away from its starting point. This atmosphere is ***unstable***.
  - if $\Gamma_d > \gamma$ then the acceleration is negative, and the parcel will accelerate downward, toward its starting point. This atmosphere is ***stable***.

- Potential temperature can also be used to assess the static stability of unsaturated air.
  - Assume a parcel starts out at $z = 0$ and has the same potential temperature as its environment, $\theta'(0) = \theta(0) = \theta_0$.
  - If the parcel is lifted a distance z its potential temperature remains at $\theta'(z) = \theta_0$, since potential temperature is conserved in adiabatic motion. However, the environment's temperature at the new altitude is given by the MacLaren series expansion

$$\theta(z) = \theta_0 + \frac{\partial \theta}{\partial z} z + \frac{1}{2!}\frac{\partial^2 \theta}{\partial z^2} z^2 + \frac{1}{3!}\frac{\partial^3 \theta}{\partial z^3} z^3 + \cdots . \tag{18}$$

  - If we limit ourselves to small displacements we can truncate the series (18) after two terms and using this in (13) gives the acceleration on the air parcel as

$$a'_z = -\frac{g}{\theta}\frac{\partial \theta}{\partial z} z. \tag{19}$$

- From (19) we see that:
  - if $\frac{\partial \theta}{\partial z} < 0$ then the acceleration is positive, and the parcel will accelerate upward, away from its starting point. This atmosphere is ***unstable***.
  - if $\frac{\partial \theta}{\partial z} > 0$ then the acceleration is negative, and the parcel will accelerate downward, toward its starting point. This atmosphere is ***stable***.

An Important Differential Equation to Know

- In meteorology we often encounter second-order ordinary differential equations (ODE) of the form
$$\frac{d^2y}{dt^2} + \beta y = 0, \tag{20}$$
where β is either a positive or negative real number.

- You should know the solutions to this for $\beta > 0$ and $\beta < 0$.
  - If $\beta > 0$ then equation (20) has the solution
$$y(t) = A\cos(\sqrt{\beta}t) + B\sin(\sqrt{\beta}t). \tag{21}$$
  - If $\beta < 0$ then equation (20) has the solution
$$y(t) = A\exp(\sqrt{|\beta|}t) + B\exp(-\sqrt{|\beta|}t), \tag{22}$$
where $|\beta|$ indicates the absolute value of β.

- If you forget which solution goes to which equation, simply pick a solution and substitute it into the equation. If it works, it is the solution. If it doesn't work, it isn't.

Brünt-Väisala Frequency

- The parcel's vertical acceleration is $a'_z = d^2z/dt^2$. Therefore, (19) can be written as
$$\frac{d^2z}{dt^2} + \frac{g}{\theta}\frac{\partial\theta}{\partial z}z = 0,$$
or
$$\frac{d^2z}{dt^2} + N^2z = 0, \tag{23}$$
where
$$N^2 = \frac{g}{\theta}\frac{\partial\theta}{\partial z}. \tag{24}$$

- If $\partial\theta/\partial z > 0$ (stable atmosphere) then $N^2 > 0$ and the solution to (23) has the form of (21). The solution for the parcel's displacement with time is then
$$z(t) = A\cos(Nt) + B\sin(Nt). \tag{25}$$
  - In a stable atmosphere a displaced parcel will oscillate up and down at an angular frequency of N.
  - N is called the ***Brünt-Väisala frequency***.
  - The more stable the atmosphere the larger (faster) the Brünt-Väisala frequency.
  - In a stable atmosphere the Brünt-Väisala frequency is a real number.

- N from (24) is an ***angular frequency***, and has units of radians per second.
- To convert the units of N to ***natural frequency*** (cycles per second or Hz) you divide by 2π.

- If $\partial\theta/\partial z < 0$ (unstable atmosphere) then $N^2 < 0$ and theh solution to (23) has the form of (22). The solution for the parcel's displacement with time is then[2]

$$z(t) = Ae^{|N|t} + Be^{-|N|t}. \tag{26}$$

  - In an unstable atmosphere a displaced parcel will not oscillate. Instead, it will accelerate exponentially away from its starting point.
  - The more unstable the atmosphere the larger (higher) the parcel will accelerate away.
  - In an unstable atmosphere the Brünt-Väisala frequency is an imaginary number.

- The following table summarizes the different ways of quantifying the static stability of a dry atmosphere.

	γ	$\partial\theta/\partial z$	N^2	N
Stable	$< \Gamma_d$	> 0	> 0	real
Unstable	$> \Gamma_d$	< 0	< 0	imaginary
Neutral	Γ_d	0	0	0

Static Stability of Saturated Air

- If an air parcel is saturated, then when it is lifted it will cool at the moist-adiabatic lapse rate.
- This complicates stability, because we now have to assess what happens to both a saturated and an unsaturated air parcel.
- The conditions are:

Absolutely stable	$\Gamma_d > \Gamma_s > \gamma$	Both parcels are stable.
Conditionally stable/unstable	$\Gamma_d > \gamma > \Gamma_s$	Saturated parcel is unstable.
Absolultely unstable	$\gamma > \Gamma_d > \Gamma_s$	Both parcels are unstable.

[2] Note that $\sqrt{|N^2|} = |N|$, and that the absolute value of an imaginary number is simply its magnitude.

Lifting and Stability Change

- Adiabatic lifting or sinking of an entire layer of air can change its stability.
- For an *unsaturated* layer that remains unsaturated, the stability change depends on the initial stability of the the layer.
  - Lifting moves lapse rate away from neutrality.
  - Sinking moves lapse rate toward neutrality.
- This is summarized in the following table:

	Initially stable	Initially unstable	Initially neutral
Lifting	less stable	more stable	no change
Sinking	less unstable	more unstable	no change

- The effect of lifting or sinking on stability is best understood by plotting a layer on a Skew-T diagram and watching what happens to the lapse rate as it is lifted or sunk adiabatically.
- Since the atmosphere is rarely ever unstable with respect to unsaturated parcels, lifting of an unsaturated layer always decreases stability, while sinking (***subsidence***) always increases stability.
- This is how ***subsidence inversions*** form.

Convective Instability

- If a layer of air is lifted until the entire layer is saturated, the stability of the layer will depend on how the *equivalent potential temperature*, or alternately the *wet-bulb potential temperature* changes with height. The conditions are:

Convectively stable	$\frac{\partial\theta_e}{\partial z} > 0$	$\frac{\partial\theta_w}{\partial z} > 0$
Convectively unstable	$\frac{\partial\theta_e}{\partial z} < 0$	$\frac{\partial\theta_w}{\partial z} < 0$
Convectively neutral	$\frac{\partial\theta_e}{\partial z} = 0$	$\frac{\partial\theta_w}{\partial z} = 0$

- Note that a layer of unsaturated air that is initially stable may become unstable by lifting.

Autoconvective Lapse Rate

- In the prior discussion of instability the air parcel needed to be moved from its initial point in order for it to accelerate away.
- If the lapse rate is steep enough the atmosphere will overturn on its own, without needing an initial nudge or impetus.
- The lapse rate at which this occurs is called the ***autoconvective lapse rate***.
- The autoconvective lapse rate is derived as follows:
  - The autoconvective lapse rate occurs when the density of the air increases with height,
  $$\frac{\partial \rho}{\partial z} > 0. \tag{27}$$
  - Differentiate the ideal gas law with respect to z,
  $$\frac{\partial p}{\partial z} = R_d T \frac{\partial \rho}{\partial z} + R_d \rho \frac{\partial T}{\partial z},$$
  and rearrange to get
  $$\frac{\partial \rho}{\partial z} = \frac{1}{R_d T}\frac{\partial p}{\partial z} - \frac{\rho}{T}\frac{\partial T}{\partial z}. \tag{28}$$
  - Comparing (28) with (27) we get
  $$\frac{1}{R_d T}\frac{\partial p}{\partial z} - \frac{\rho}{T}\frac{\partial T}{\partial z} > 0,$$
  or
  $$-\frac{\partial T}{\partial z} > -\frac{1}{R_d \rho}\frac{\partial p}{\partial z}. \tag{29}$$
  - From the hydrostatic equation we can substitute $-\rho g$ for $\partial p/\partial z$ in (29) to get
  $$-\frac{\partial T}{\partial z} > -\frac{1}{R_d \rho}(-\rho g) = \frac{g}{R_d}, \tag{30}$$
  and therefore we get
  $$\gamma_{auto} = \frac{g}{R_d}. \tag{31}$$
- The autoconvective lapse rate has a value of 34.2 K km^{-1}.
- Lapse rates this large only occur in the surface layer, within a few meters of the ground, during very intense solar heating.
- Autoconvective lapse rates are responsible for the formation of ***dust devils***.