The Vorticity Equation and Conservation of Angular Momentum

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Abstract

The link between convergence and absolute angular momentum in the production of vorticity is explored by deriving the barotropic potential vorticity equation directly from the principle of conservation of absolute angular momentum. The derivation is relatively simple, and though not a substitute for traditional derivations of the vorticity equation, it better illustrates that the physical basis for the divergence term is the conservation of absolute angular momentum.

Background

Sometime during their first course in dynamic meteorology every university-trained meteorologist encounters a derivation of the synoptic-scale vorticity equation,

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\nabla \bullet \vec{V} \quad . \tag{1}$$

This equation states that for large-scale atmospheric motions absolute vorticity (the sum of relative vorticity, ζ, and planetary vorticity, f) is created or destroyed solely through convergence and divergence. For a barotropic fluid of variable depth h, Eqn. (1) becomes

$$\frac{D}{Dt}(\zeta + f) = (\zeta + f)\frac{D(\ln h)}{Dt} \quad . \tag{2}$$

In this form of the vorticity equation (called the barotropic potential vorticity equation) the divergence term is manifested through changes in the depth of the fluid.

The physical principle behind the divergence term in either Eqn. (1) or (2) is the conservation of absolute angular momentum (Holton, 2004), which is the angular momentum as viewed from an observer fixed in space. However, this physical basis is not obvious from either the equations or their derivation. The connection between divergence and change in absolute vorticity change can be illustrated directly and simply by deriving Eqn. (2) directly from a mathematical statement of the conservation of absolute angular momentum. Deriving the equation in this way is not a substitute for the traditional derivation, but is useful for convincing students that the physical meaning of the divergence term is solely the conservation of absolute angular momentum. This derivation is illustrated in the following sections.

Absolute Angular Momentum

An axisymmetric column of fluid rotating at a fixed point on the Earth's surface has two contributions to its absolute angular momentum. One is due to its motion around the Earth's axis of rotation (orbital angular momentum), and the other due to its spin around the vertical axis through its center of mass (spin angular momentum) (Corben and Stehle, 1994). Viewed from space there are two external forces excluding friction acting on the column. These are gravitational and pressure gradient force. The gravitational force passes through the centers of mass of both the fluid column and the Earth and does not exert a torque that can alter either the orbital or spin angular momentum. Likewise, the horizontal pressure gradient force passes through the center of rotation of the column, and is axisymmetric, and also affects neither component of the angular momentum. The orbital and spin angular momentums are essentially uncoupled and independent.

Angular momentum is a vector quantity, and the spin angular momentum is oriented along the axis of rotation of the column. Since the fluid column remains at a fixed point on Earth and is at all times oriented to the local vertical, the spin-angular momentum cannot be constant, because viewed from space its direction is changing as the Earth rotates. The change in the spin angular momentum vector, $\vec{L}_S$, as viewed in an absolute frame versus a reference frame attached to the surface of the Earth (and therefore rotating with angular velocity $\vec{\Omega}_e$) is given by

$$\frac{D_a \vec{L}_S}{Dt} = \frac{D\vec{L}_S}{Dt} + \vec{\Omega}_e \times \vec{L}_S \,, \tag{3}$$

where the derivative having the subscript *a* is with respect to an absolute or inertial reference frame, while the derivative without a subscript is in an Earth-relative reference frame. The cross-product term in Eqn. (3) is always perpendicular to $\vec{L}_S$ and cannot change the magnitude of the angular momentum in either reference frame. We can therefore safely write the scalar conservation equation in the Earth-relative frame,

$$\frac{DL_S}{Dt} = 0 \,, \tag{4}$$

where L_S is the magnitude of the vector $\vec{L}_S$.

Derivation of the Barotropic Potential Vorticity Equation

Taking the fluid column to be infinitesimally skinny allows an approximation of solid-body rotation (Brown, 1991). The magnitude of the spin angular momentum of the column is then

$$L_S = I\Omega_S \,, \tag{5}$$

where I is the moment of inertia and Ω_S is its angular speed. The angular speed can be decomposed into an Earth-relative angular speed, Ω_r, and that component of the Earth's angular velocity in the local vertical, denoted as Ω_{ek} so that

$$\Omega_S = \Omega_r + \Omega_{ek} \,. \tag{6}$$

Eqns. (4), (5), and (6) result in

$$\frac{D}{Dt}\left(\Omega_r + \Omega_{ek}\right) = -\left(\Omega_r + \Omega_{ek}\right)\frac{D\left(\ln I\right)}{Dt} \,. \tag{7}$$

In solid-body rotation the vorticity is twice the angular velocity, allowing the relative and Earth angular velocities to be written in terms of the relative and planetary vorticities,

$$\Omega_r = \zeta/2$$
$$\Omega_{ek} = f/2$$

respectively. This turns Eqn. (7) in to

$$\frac{D}{Dt}\left(\zeta + f\right) = -\left(\zeta + f\right)\frac{D\left(\ln I\right)}{Dt} \,. \tag{8}$$

All that remains, in order to recover Eqn. (2), is to show that

$$\frac{D\left(\ln I\right)}{Dt} = -\frac{D\left(\ln h\right)}{Dt} \,. \tag{9}$$

The moment of inertia for the column is $MR^2/2$ (Halliday and Resnick, 1981) where M is the mass and R the radius, so we have

$$\frac{D\left(\ln I\right)}{Dt} = \frac{2}{R}\frac{DR}{Dt} = \frac{2U}{R} \,, \tag{10}$$

where U is the radial velocity of the particles on the periphery. The volume of a the column is $\pi R^2 h$, and remains constant as the radius changes. From this comes

$$\frac{D}{Dt}\left(\pi R^2 h\right) = \pi\left(2hR\frac{DR}{Dt} + R^2\frac{Dh}{Dt}\right) = 0\,, \tag{11}$$

which rearranges to

$$\frac{D\left(\ln h\right)}{Dt} = -\frac{2U}{R}\,, \tag{12}$$

and with Eqn. (10) establishes the validity of Eqn. (9). Using Eqn. (9) in Eqn. (8) results in

$$\frac{D}{Dt}(\zeta + f) = (\zeta + f)\frac{D\left(\ln h\right)}{Dt}\ , \tag{2}$$

and shows that the barotropic potential vorticity equation can be derived directly from the principle of conservation of angular momentum.

Conclusion

The barotropic potential vorticity equation can be derived directly from the principle of conservation of absolute angular momentum. The derivation is not a substitute for the traditional derivations of the vorticity equation. It is useful, however, in demonstrating that the physical basis of the divergence term in the vorticity equation is due solely to conservation of absolute angular momentum, something that is not clear from traditional derivations of the equation.

References

Brown, R.A., 1991: *Fluid Mechanics of the Atmosphere*, Academic Press, 489 pp.
Corben, H.C. and P. Stehle, 1994: *Classical Mechanics* (2nd ed.), Dover, 389 pp.
Halliday, D. and R. Resnick, 1981: *Fundamentals of Physics* (2d ed.), J. Wiley & Sons, 816 pp.
Holton, J.R., 2004: *An Introduction to Dynamic Meteorology*, Elsevier, 535 pp.