The Monty Hall (Game Show) Problem Solved Using Bayes' Theorem

The Problem

There are three doors (A, B, and C), behind one of which is the prize. The prize is randomly placed behind one of the three doors. After you choose a door, the host always opens one of the other two doors showing that the prize is not behind the door revealed. You are then given the opportunity to stick with your original choice of door, or switch to the other door. What should you do to have the best odds of winning the prize?

The Setup

You select Door A and the host reveals Door B.

- $\tilde{A}$ represents the event that you chose Door A
- A represents the event that the prize is behind Door A
- B represents the event that the prize is behind Door B
- C represents the event that the prize is behind Door C
- $\hat{B}$ represents the event that the host reveals door B

Bayes' theorem for the probability that the prize is behind Door A given that the host reveals Door B and that you chose Door A is:

$$P(A \mid \hat{B}, \tilde{A}) = \frac{P(\hat{B} \mid A, \tilde{A}) P(A, \tilde{A})}{P(\hat{B}, \tilde{A})} \tag{1}$$

Note that $\tilde{A}$ appears because all our analyses are based on the condition that you chose Door A. Since it appears in every one of our probabilities, we will not write it explicitly. However, <u>we must remember that our analysis assumes that we have already chosen Door A</u>! So, we can write

$$P(A \mid \hat{B}) = \frac{P(\hat{B} \mid A) P(A)}{P(\hat{B})} \tag{2}$$

and similarly for the probabilities that the prize is behind the other doors given that we chose A and the host revealed B,

$$P(B \mid \hat{B}) = \frac{P(\hat{B} \mid B) P(B)}{P(\hat{B})} \tag{3}$$

$$P(C \mid \hat{B}) = \frac{P(\hat{B} \mid C) P(C)}{P(\hat{B})} \tag{4}$$

where the total probability of the host revealing Door B is given by

$$P(\hat{B}) = P(\hat{B} \mid A)P(A) + P(\hat{B} \mid B)P(B) + P(\hat{B} \mid C)P(C) \tag{5}$$

The Probabilities

The prize is randomly placed, so the probabilities for each door are each 1/3,

$$P(A) = P(B) = P(C) = 1/3 \tag{6}$$

For the conditional probabilities below, remember that <u>an unstated but implicit condition is that you have already chosen Door A</u>.

- If the prize is behind A, then the host will only open either door B or C, so

$$P(\hat{B} \mid A) = 1/2 \tag{7}$$

- If the prize is behind B, then the host will never open door B, so

$$P(\hat{B} \mid B) = 0 \tag{8}$$

- If the prize is behind C, then the host will only open door B, so

$$P(\hat{B} \mid C) = 1 \tag{9}$$

Using the values from (6) thru (9) in (5), we get

$$P(\hat{B}) = 1/2 \tag{10}$$

Thus, we obtain the following probabilities from (2) thru (4), for the prize being behind each door given that you chose Door A and the host reveals Door B:

$$P(A \mid \hat{B}, \tilde{A}) = 1/3 \tag{11}$$
$$P(B \mid \hat{B}, \tilde{A}) = 0 \tag{12}$$
$$P(C \mid \hat{B}, \tilde{A}) = 2/3 \tag{13}$$

We have written $\tilde{A}$ to emphasize that everything is based on our having chosen Door A.

Thus, you are always better off switching your choice once the host reveals the empty door!