<u>Mean Distance Between Cloud Droplets</u>

References: *The Art of Probability for Scientists and Engineers*, R.W. Hamming, Westview Press, 1991
Microphysics of Clouds and Precipitation, H.R. Pruppacher and J.D. Klett, Kluwer Acad. Pub., 1997

Probability of Finding *x* Number of Drops in a Small Sub-volume

Imagine a volume (V_0) containing a uniformly random distribution of a total of N cloud droplets, with mean number density, $n = N/V_0$. The probability of finding x number of droplets in a given sub-volume, V, is found from the binomial distribution,

$$P(x) = \frac{N!}{(N-x)!x!} p^x (1-p)^{N-x} .$$

In this expression, p is the probability of finding a ***specific*** droplet in the sub-volume, and is just the ratio of the sub-volume to the total volume,

$$p = V/V_0 .$$

If the sub-volume is very small compared to the total volume, then p is also very small. In this case (assuming that N is large), it is appropriate to use the Poisson distribution, which is the limiting form of the binomial distribution for large N and small p. Therefore, for small sub-volumes, the probability of finding x droplets in the sub-volume is given by

$$P(x) = \frac{e^{-\mu} \mu^x}{x!} , \quad \textbf{(1)}$$

where μ is the mean number of droplets in the sub-volume. The mean number of droplets in a sub-volume is simply

$$\mu = nV . \quad \textbf{(2)}$$

The Probability Density Function for the Nearest Neighbor to a Droplet

The probability that the nearest neighbor to a droplet lies in the interval between r and $r + dr$ [denoted $dP_{nn}(r)$], is a compound probability, given by the product of the probability that there are no droplets within a spherical volume of radius r centered on the droplet, and the probability that there is at least one droplet within a spherical shell of inner radius r, and outer radius, $r + dr$.

$$dP_{nn}(R) = P_{0; r \le R} \times P_{\ge 1; R \le r < R+dr} . \quad \textbf{(3)}$$

The probability that there are no droplets within a spherical volume of radius r centered on the droplet comes straight from equations (1) and (2) as follows:

$$P_{0; r \le R} = e^{-\mu} = \exp(-nV) = \exp\left(-\frac{4}{3}\pi n r^3\right) . \quad \textbf{(4)}$$

The probability of finding *at least* one droplet within a spherical shell of inner radius r, and outer radius, $r + dr$ is found as follows:

$$P_{x \geq 1} = 1 - P(0) = 1 - e^{-\mu}.$$

Since the spherical shell is infinitesimally small, the mean number of drops is going to much less than 1 (because the probability of an individual drop appearing in the sub-volume is also infinitesimally small). Therefore, $\mu << 1$, and we can approximate

$$e^{-\mu} \cong 1 - \mu,$$

so that

$$P_{x \geq 1} \cong \mu = n\,dV = 4\pi\, n\, r^2 dr. \qquad \textbf{(5)}$$

Using (4) and (5) in (3) we get that

$$dP_{nn}(r) = 4\pi\, n\, r^2 \exp\left(-\frac{4}{3}\pi\, n\, r^3\right) dr.$$

Since

$$dP_{nn}(r) = p_{nn}(r) dr,$$

the probability density function for the nearest neighbor distance is

$$p_{nn}(r) = 4\pi\, n\, r^2 \exp\left(-\frac{4}{3}\pi\, n\, r^3\right). \qquad \textbf{(6)}$$

The Mean Distance Between Droplets

Once we have the probability density function for the nearest neighbor distance, we can find the mean distance via

$$\bar{r} = \int_0^\infty r\, p_{nn}(r) dr = 4\pi\, n \int_0^\infty r^3 \exp\left(-\frac{4}{3}\pi\, n\, r^3\right) dr.$$

By defining a variable transformation as

$$t = \frac{4}{3}\pi\, n\, r^3$$

the integral becomes

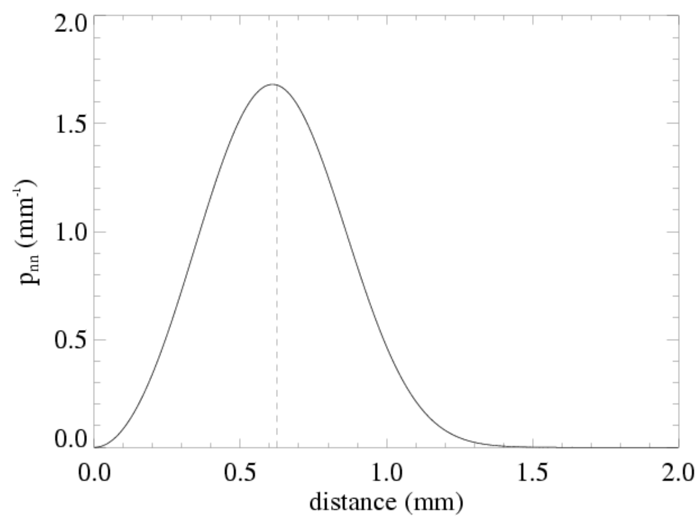
$$\bar{r} = \left(\frac{3}{4\pi\, n}\right)^{1/3} \int_0^\infty t^{1/3} \exp(-t) dt = \left(\frac{3}{4\pi\, n}\right)^{1/3} \Gamma(4/3)$$

where Γ is the gamma function. The value of $\Gamma(4/3)$ is 0.89338, so the mean distance between droplets is

$$\bar{r} = 0.554 n^{-1/3}.$$

The graph below shows the probability density function for n = 700 cm^{-3}, with the location of the mean shown by the dashed line. The probability density is skewed, so that the mean is not at the same location as the mode. They are close, however, as the mode is at

$$r_m = 0.549 n^{-1/3}.$$



Probability density function for the nearest-neighbor distance, for n = 700 cm^{-3}. Dashed line shows position of mean.