The Coriolis and Centrifugal Torques in Gyroscopic Precession

Alex J. DeCaria

I. INTRODUCTION

The torque required on an axisymmetric gyro in order to maintain steady precession is[1]

$$\vec{\tau} = \left[C + \left(C - A \right) \frac{\Omega}{\omega} \cos\varphi \right] \vec{\Omega} \times \vec{\omega} \tag{1}$$

where $\vec{\Omega}$ is the angular velocity of the precession, $\vec{\omega}$ is the spin angular velocity of the gyro, φ is the angle between $\vec{\Omega}$ and $\vec{\omega}$, and C and A are the moments of inertia around the spin axis of the gyro and around a line normal to the spin axis of the gyro respectively. A gyro conserving angular momentum with respect to space will appear to an observer on Earth to undergo steady precession with precessional angular velocity equal and opposite to the angular velocity of the Earth, $\vec{\Omega}_e$. For this case Eq. (1) becomes

$$\vec{\tau} = -\left[C - \left(C - A \right) \frac{\Omega_e}{\omega} \cos\varphi \right] \vec{\Omega}_e \times \vec{\omega}. \tag{2}$$

The torque in Eq. (2) must be due to the combined effects of both the Coriolis and centrifugal forces, since these are the only forces capable of creating a torque in the observer's Earth-bound reference frame. Examining which terms in Eq. (2) are due to the Coriolis force, and which are due to the centrifugal force, is a thought-provoking exercise that can help students gain a better understanding of both gyroscopic motion and apparent forces in rotating reference frames.

In Sec. II the general formulas for separately calculating the Coriolis and centrifugal torques on a gyro of arbitrary shape are derived. In Sec. III these formulas are used to calculate the Coriolis and centrifugal torques for an axisymmetric gyro consisting of a spinning ring of mass. Section IV discusses the results obtained for the spinning ring of mass, and shows that the two terms in Eq. (2) that contain C are the Coriolis torque terms, while the single term that contains A is due to the centrifugal torque. The reason for the existence of two terms for the Coriolis torque is also explained. Results for a spherically symmetric gyro are then presented and discussed in Sec. V, with the interesting result that for a spherically symmetric gyro the centrifugal torque exactly balances one of the Coriolis torque terms. Section VI provides a summary and some suggested problems for students.

II. THE CORIOLIS AND CENTRIFUGAL TORQUES

For any generally-shaped gyro the Coriolis force on a point of mass dm on the gyro is

$$\vec{F}_{Cor} = -2\left(\vec{\Omega}_e \times \vec{V} \right) dm, \tag{3}$$

where $\vec{V}$ is the velocity of the point relative to the Earth. The contribution to the total torque on the gyro contributed by this force is

$$d\vec{\tau}_{Cor} = -2\vec{r} \times \left(\vec{\Omega}_e \times \vec{V} \right) dm, \tag{4}$$

where $\vec{r}$ is the position vector from the gyro's center of mass. There are two contributions to the velocity: that due to the spin of the object, and that due to the precession. The total of these contributions is

$$\vec{V} = \vec{\omega}\times\vec{r} - \vec{\Omega}_e \times \vec{r} = \left(\vec{\omega} - \vec{\Omega}_e\right)\times\vec{r} \tag{5}$$

so that

$$d\vec{\tau}_{Cor} = -2\vec{r}\times\left(\vec{\Omega}_e \times\left[\left(\vec{\omega} - \Omega_e\right)\times\vec{r}\,\right]\right)dm\,, \tag{6}$$

which after application of vector identities becomes

$$d\vec{\tau}_{Cor} = -2\left(\vec{\Omega}_e \bullet \vec{r}\right)\left[\vec{r}\times\left(\vec{\omega} - \vec{\Omega}_e\right)\right]dm\,. \tag{7}$$

The torque due to the Coriolis force on the gyro is the integral of Eq. (7) over the entire body.

The centrifugal force at a point of mass, *dm*, on the body is

$$\vec{F}_{cen} = \Omega_e^2 \vec{R}\,dm\,, \tag{8}$$

where $\vec{R}$ is the perpendicular distance to a line parallel to the Earth's axis of rotation through the center of mass of the body. This produces a torque given by

$$d\vec{\tau}_{cen} = \vec{r}\times\left(\Omega_e^2 \vec{R}\right)dm\,, \tag{9}$$

and the torque due to the centrifugal force on the gyro is the integral of Eq. (9) over the entire body.

III. CALCULATION OF TORQUES FOR A SPINNING RING

The calculation of the Coriolis and centrifugal torques for an axisymmetric gyro consisting of a ring of radius r_0 proceeds as follows. A suitable coordinate system for the integration is a right-handed coordinate system with the *z*-coordinate in the direction of $\vec{\omega}$ and the *x*-coordinate in the direction of $\vec{\Omega}_e \times \vec{\omega}$ (Fig. 1). In this coordinate system the appropriate vectors have components

$$\begin{aligned}
\vec{r} &= r_0\left(\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)\\
\vec{R} &= r_0\left(\sin\theta\,\hat{i} + \cos^2\varphi\cos\theta\,\hat{j} - \cos\varphi\sin\varphi\cos\theta\,\hat{k}\right)\\
\vec{\Omega}_e &= \Omega_e\left(\sin\varphi\,\hat{j} + \cos\varphi\,\hat{k}\right)\\
\vec{\omega} &= \omega\,\hat{k}
\end{aligned} \tag{10.a-d}$$

where θ is the angle between $\vec{r}$ and the *y* axis (measured clockwise), and φ is the angle between the two angular velocity vectors. The incremental mass, *dm*, is

$$dm = lRd\theta \tag{11}$$

where *l* is the mass per unit length of the ring. Using Eqs. (10) and (11) in Eqs. (7) and (9) and integrating results in the following equations for the Coriolis and centrifugal torques:

$$\vec{\tau}_{Cor} = -2l\Omega_e r_0^3 \sin\varphi \int_0^{2\pi} \begin{bmatrix} (\omega - \Omega_e \cos\varphi)\cos^2\theta\, \hat{i} \\ -(\omega - \Omega_e \cos\varphi)\cos\theta \sin\theta\, \hat{j} \\ -\Omega_e \sin\varphi \cos\theta \sin\theta\, \hat{k} \end{bmatrix} d\theta \tag{12}$$

and

$$\vec{\tau}_{cen} = -l\Omega_e^2 r_0^3 \int_0^{2\pi} \begin{bmatrix} \sin\varphi \cos\varphi \cos^2\theta\, \hat{i} \\ -\sin\varphi \cos\varphi \cos\theta \sin\theta\, \hat{j} \\ +\left(1-\cos^2\varphi\right)\cos\theta \sin\theta\, \hat{k} \end{bmatrix} d\theta . \tag{13}$$

Owing to the orthogonality of sines and cosines , only the $\hat{i}$ components of these equations will be non-zero on integration, and the resulting equations are

$$\vec{\tau}_{Cor} = -Mr_0^2 \Omega_e \sin\varphi (\omega - \Omega_e \cos\varphi)\, \hat{i} \tag{14}$$

and

$$\vec{\tau}_{cen} = -(M/2)\, r_0^2 \Omega_e^2 \sin\varphi \cos\varphi\, \hat{i} \tag{15}$$

where M is the total mass of the ring ($M = 2l\pi r_0$). The moments of inertia for the ring of mass are

$$\begin{aligned} A &= Mr_0^2/2 \\ C &= Mr_0^2 \end{aligned} \tag{16a and b}$$

so Eqs. (14) and (15) can be written as

$$\vec{\tau}_{Cor} = -C\Omega_e \sin\varphi (\omega - \Omega_e \cos\varphi)\, \hat{i} \tag{17}$$

and

$$\vec{\tau}_{cen} = -A\Omega_e^2 \sin\varphi \cos\varphi\, \hat{i} . \tag{18}$$

Summing Eqs. (17) and (18) yields

$$\vec{\tau} = -\left[C - (C - A)\frac{\Omega_e}{\omega}\cos\varphi \right] \Omega_e \omega \sin\varphi\, \hat{i} , \tag{19}$$

which can be written as

$$\vec{\tau} = -\left[C - (C - A)\frac{\Omega_e}{\omega}\cos\varphi \right] \vec{\Omega}_e \times \vec{\omega} . \tag{20}$$

Equation (20) is identical to Eq. (2) as expected.

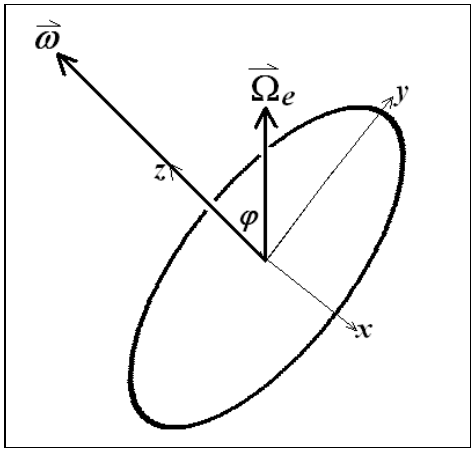

Fig. 1 – Coordinate system with x axis in the direction of $\vec{\Omega}_e \times \vec{\omega}$ and the z axis in the direction of $\vec{\omega}$. The angle between $\vec{\Omega}_e$ and $\vec{\omega}$ is φ, and the azimuth angle, θ, is measured clockwise from the y axis.

IV. DISCUSSION OF RESULTS FOR SPINNING RING

Examination of Eqs. (17), (18) and (20) shows that the terms that involve the moment of inertia (C) around the spin axis are due to the Coriolis torque, and the term involving the moment of inertia around an axis perpendicular to the spin axis (A) is due to the centrifugal torque. That there are two Coriolis terms at first seems puzzling, but is explained by the fact that the velocity at a point on the gyro can be split into a part due to the spin of the gyro and a part due to the precession, and there is a separate Coriolis torque for each of these velocities.

The first term in Eq. (20),

$$\vec{\tau}_1 = -C\,\vec{\Omega}_e \times \vec{\omega}, \qquad (21)$$

is the torque associated with that part of the Coriolis force due to the spin of the gyro. The second term,

$$\vec{\tau}_2 = \left(C\frac{\Omega_e}{\omega}\cos\varphi \right)\vec{\Omega}_e \times \vec{\omega}, \qquad (22)$$

is associated with that part of the Coriolis force due to the precession of the gyro. The final term,

$$\vec{\tau}_3 = -\left(A\frac{\Omega_e}{\omega}\cos\varphi \right)\vec{\Omega}_e \times \vec{\omega}, \qquad (23)$$

is attributed to the centrifugal force. Notice that in component form the magnitude of the spin angular velocity, ω, cancels in both Eq. (22) and (23), which is expected, since physically neither of these terms should depend on the rate of spin, but only on the rate precession.

A qualitative understanding of these three terms follows. The Coriolis force at a point on the ring is proportional to and opposite of the cross product of the angular velocity of the Earth and the Earth-relative velocity of the point,

$$\vec{F}_{Cor} \propto -\vec{\Omega}_e \times \vec{V} \; . \tag{24}$$

The velocity is due to a combination of spin and precession. Figure 2.a shows the direction of the velocity and Coriolis force due to spin only at two points on the ring. Though only drawn for two points, consideration of Eq. (24) for any other points on the ring will serve to convince that the net result of Coriolis force due to spin is a negative torque around the *x* axis. Careful thought will also result in realizing that this torque will disappear if $\vec{\omega}$ and $\vec{\Omega}_e$ are either parallel or opposite. Figure 2.b is the same as Fig. 2.a, only for the Coriolis force due to the precession of the gyro. The forces in this case are opposite to those of Fig 2.a (recall that the precession angular velocity is opposite to $\vec{\Omega}_e$), and so the torques in the two cases are opposite. This is reflected in the opposite signs in Eqs. (21) and (22). As with the spin Coriolis torque, the torque from the precessional Coriolis force disappears if $\vec{\omega}$ and $\vec{\Omega}_e$ are either parallel or opposite, but in addition, it disappears if the two vectors are also perpendicular.

The centrifugal force is everywhere oriented normal to, and away from, the line on which $\vec{\Omega}_e$ lies, and the two force arrows drawn in Fig. 2.c show direction of the centrifugal force at the same two points used in the prior figures. The centrifugal forces create a torque that has the same orientation as that due to the Coriolis force from the spin, as confirmed by the same signs in Eqs. (21) and (23).

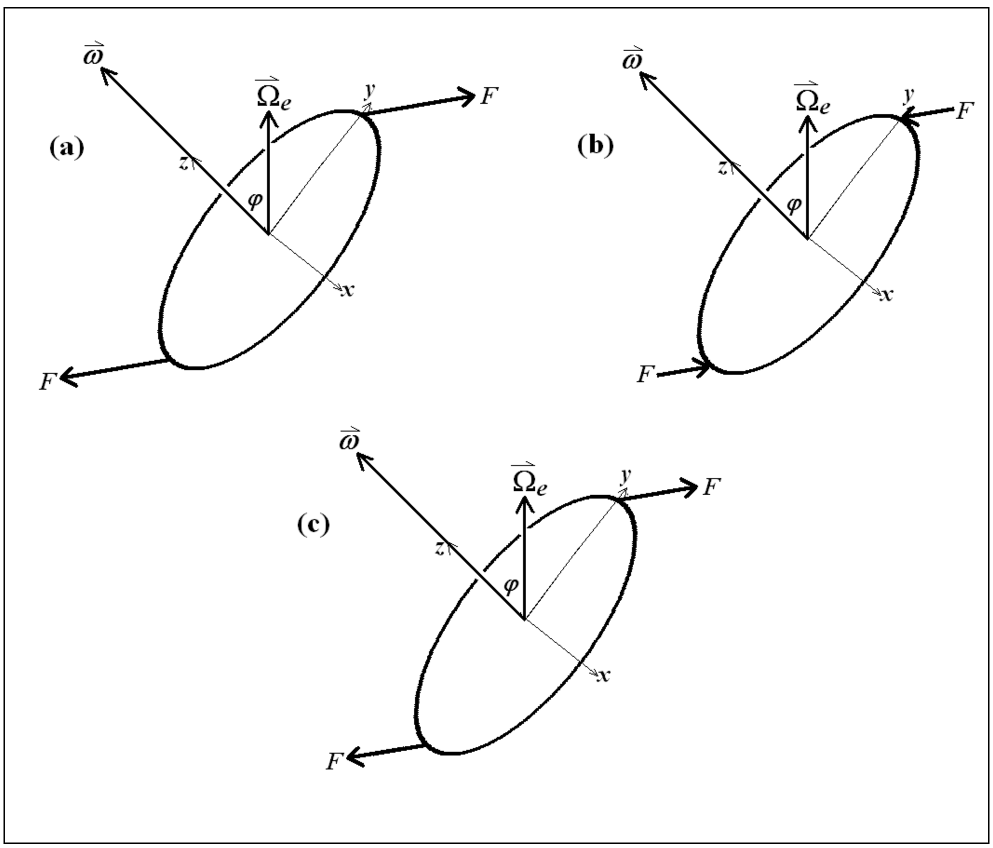

Fig. 2 – Arrows marked *F* show the directions of the forces at two points for the torques due to (a) Coriolis force from spin; (b) Coriolis force from precession; and (c) centrifugal force.

V. RESULTS AND DISCUSSION FOR A SPHERICAL SHELL

For a spherically symmetric gyro a thin spherical shell of radius r_0 is chosen. In this case the relevant vectors in component form are

$$\vec{r} = r_0\left(\sin\phi\sin\theta\,\hat{i} + \sin\phi\cos\theta\,\hat{j} + \cos\phi\,\hat{k}\right)$$
$$\vec{\Omega}_e = \Omega_e\left(\sin\varphi\,\hat{j} + \cos\varphi\,\hat{k}\right) \qquad (25.a\text{-}c)$$
$$\vec{\omega} = \omega\,\hat{k}$$

where ϕ is the angle from the z axis. The vector $\vec{R}$ in this case is found by using the following two expressions which come from Fig. 3,

$$\vec{a} + \vec{R} = \vec{r}$$
$$\vec{a} \bullet \vec{R} = 0 \qquad (26)$$

which represent two equations and two unknowns, $\vec{a}$ and $\vec{R}$. Writing these expressions in component form and eliminating a yields the following expression for $\vec{R}$,

$$\vec{R} = r_0\begin{bmatrix} \sin\phi\sin\theta\,\hat{i} \\ +\left(\cos^2\varphi\sin\phi\cos\theta - \cos\varphi\sin\varphi\cos\phi\right)\hat{j} \\ +\left(\sin^2\varphi\cos\phi - \cos\varphi\sin\varphi\sin\phi\cos\theta\right)\hat{k} \end{bmatrix}. \qquad (27)$$

Using Eqs. (25) and (27) in Eqs. (7) and (9) and integrating is complex due to the shear number of terms involved. However, many of the terms are zero on integration due to the orthogonality of sine and cosine functions, so that any term whose dependence on θ is through a single $\sin\theta$ or $\cos\theta$ term, or a product of $\sin\theta$ or $\cos\theta$ will disappear. Some other identities that are useful in the integration are

$$\int_0^\pi \sin^3\phi\,d\phi = 4/3$$
$$\int_0^\pi \cos^3\phi\,d\phi = 0$$
$$\int_0^\pi \sin^2\phi\cos\phi\,d\phi = 0 \qquad (28)$$
$$\int_0^\pi \cos^2\phi\sin\phi\,d\phi = 2/3$$

and it is also helpful to know that for the spherical shell the moments of inertia are

$$C = A = 2Mr_0^2/3. \qquad (29)$$

With patience and care, integration of Eqs. (7) and (9) yields the following expressions,

$$\vec{\tau}_{Cor} = -C\left(1-\Omega_e\cos\varphi\right)\Omega_e\sin\varphi\,\hat{i}$$
$$\vec{\tau}_{cen} = -C\Omega_e^2\cos\varphi\sin\varphi\,\hat{i} \qquad (30a,b)$$

which when summed give the total torque as

$$\vec{\tau} = -C\,\vec{\Omega}_e\times\vec{\omega}. \qquad (31)$$

This is exactly what is obtained from Eq. (2) when C and A are identical, as they are for a spherically symmetric gyro. In the spherically symmetric gyro the Coriolis torque due to precession and the centrifugal torque exactly cancel, and the only remaining torque is due to the Coriolis force from the spin of the gyro.

VI. SUMMARY AND CONCLUSION

The general formulas for separately calculating the Coriolis and centrifugal torques on a gyro of arbitrary shape were derived and then used to calculate the Coriolis and centrifugal torques for an axisymmetric gyro consisting of a ring of mass, and a spherically symmetric gyro consisting of a thin, spherical shell. The results were used to show that in Eq. (2) those terms that contain C are due to the Coriolis torque while the single term that contains A is due to the centrifugal torque. The existence of two terms for the Coriolis torque was shown to be due to the splitting of the velocity into a part due to spin and a part due to precession, each of which contributes to the Coriolis force. Results for the spherically symmetric gyro showed that the centrifugal torque exactly balances one of the Coriolis torque terms in this case.

Some suggested further exercises are to perform the same derivations for other axisymmetric gyros consisting of either hollow or solid cylinders, or solid plates, and for a solid, spherically symmetric gyro.

---

[1] J. B. Scarborough, *The Gyroscope: Theory and Applications* (Interscience, New York, 1958), Chap. 3.